Programming Language Semantics

A Rewriting Approach

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2.2 Basic Computability Elements

In this section we recall very basic concepts of computability theory needed for other results in the book, and introduce our notation for these. This section is by no means intended to serve as a substitute for much thorough presentations found in dedicated computability textbooks (some mentioned in Section 2.2.4).

2.2.1 Turing Machines

Turing machines are abstract computational models used to formally capture the informal notion of a *computing system*. The *Church-Turing thesis* postulates that any computing system, any algorithm, or any program in any programming language running on any computer, can be equivalently simulated by a Turing machine. Having a formal definition of computability allows us to rigorously investigate and understand what can and what cannot be done using computing devices, regardless of what languages are used to program them. Intuitively, by a computing device we understand a piece of machinery that carries out tasks by successively applying sequences of instructions and using, in principle, unlimited memory; such sequences of instructions are today called *programs*, or *procedures*, or *algorithms*.

Turing machines are used for their theoretical value; they are not meant to be physically built. A Turing machine is a finite state device with infinite memory. The memory is very primitively organized, as one or more infinite *tapes* of cells that are sequentially accessible through *heads* that can move to the left or to the right cell only. Each cell can hold a bounded piece of data, typically a Boolean, or bit, value. The tape is also used as the input/output of the machine. The computational steps carried out by a Turing machine are also very primitive: in a state, depending on the value in the current cell, a Turing machine can only rewrite the current cell on the tape and/or move the head to the left or to the right. Therefore, a Turing machine does not have the direct capability to perform random memory access, but it can be shown that it can simulate it.

There are many equivalent definitions of Turing machines in the literature. We prefer one with a tape that is infinite at both ends and describe it next (interestingly, an almost identical machine was proposed by Emil Post independently from Alan Turing also in 1936; see Section 2.2.4). Consider a mechanical device which has associated with it a tape of infinite length in both directions, partitioned in spaces of equal size, called *cells*, which are able to hold either a 0 or an 1 and are rewritable. The device examines exactly one cell at any time, and can, potentially nondeterministically, perform any of the following four operations (or *commands*):

1. Write a 1 in the current cell;
2. Write a 0 in the current cell;
3. Shift one cell to the right;
4. Shift one cell to the left.

The device performs one operation per unit time, called a *step*. We next give a formal definition.

Definition 7. *A **(deterministic) Turing machine** $\mathcal{M}$ is a 6-tuple (Q, B, q_s, q_h, C, M), where:*

- *Q is a finite set of **internal states**;*
- *$q_s \in Q$ is the **starting state** of $\mathcal{M}$;*
- *$q_h \in Q$ is the **halting state** of $\mathcal{M}$;*
- *B is the set of **symbols** of $\mathcal{M}$; we assume without loss of generality that $B = \{0, 1\}$;*
- *$C = B \cup \{\rightarrow, \leftarrow\}$ is the set of **commands** of $\mathcal{M}$;*
- *$M : (Q - \{q_h\}) \times B \rightarrow Q \times C$ is a total function, the **transition function** of $\mathcal{M}$.*

We assume that the tape contains only 0's before the machine starts performing.

Our definition above is for *deterministic* Turing machines. One can also define nondeterministic Turing machines by changing the transition function M into a relation. Non-deterministic Turing machines have the same computational power as the deterministic Turing machines (i.e., they compute/decide the same classes of problems; computational speed or size of the machine is not a concern here), so, for our purpose in this section, we limit ourselves to deterministic machines.

A *configuration* of a Turing machine is a 4-tuple consisting of an internal state, a current cell, and two infinite strings (notice that the two infinite strings contain only 0's starting with a certain cell), standing for the cells on the left and for the cells on the right of the current cell, respectively. We let $(q, L\underline{b}R)$ denote the configuration in which the machine is in state q, with current cell b, left tape L and right tape R. For convenience, we write the left tape L backwards, that is, its head is at its right end; for example, Lb appends a b to the left tape L. Given a configuration $(q, L\underline{b}R)$, the content of the tape is LbR, which is infinite at both ends. We also let $(q, L\underline{b}R) \to_{\mathcal{M}} (q', L'\underline{b'}R')$ denote a configuration transition under one of the four commands. Given a configuration in which the internal state is q and the examined cell contains b, and if $M(q, b) = (q', c)$, then exactly one of the following configuration transitions can take place:

$(q, L\underline{b}R) \to_{\mathcal{M}} (q', L\underline{c}R)$ if $c = 0$ or $c = 1$;

$(q, L\underline{b}b'R) \to_{\mathcal{M}} (q', Lb\underline{b'}R)$ if $c = \to$;

$(q, Lb'\underline{b}R) \to_{\mathcal{M}} (q', L\underline{b'}bR)$ if $c = \leftarrow$.

Since our Turing machines are deterministic (M is a function), the relation $\to_{\mathcal{M}}$ on configurations is also deterministic. The machine starts performing in the internal state q_s. If there is no input, the initial configuration on which the Turing machine is run is $(q_s, 0^\omega\underline{0}0^\omega)$, where 0^ω is the infinite string of zeros. If the Turing machine is intended to be run on a specific input, say $x = b_1b_2\cdots b_n$, its initial configuration is $(q_s, 0^\omega\underline{0}b_1b_2\cdots b_n0^\omega)$. We let $\to^\star_{\mathcal{M}}$ denote the transitive and reflexive closure of the binary relation on configurations $\to_{\mathcal{M}}$ above.

A Turing machine $\mathcal{M}$ terminates when it reaches its halting state:

Definition 8. *Turing machine $\mathcal{M}$* ***terminates on input*** *$b_1b_2\cdots b_n$ iff*

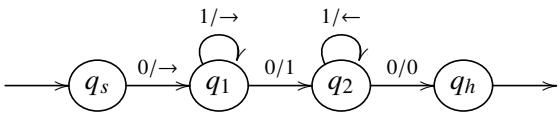
$$(q_s, 0^\omega\underline{0}b_1b_2\cdots b_n0^\omega) \to^\star_{\mathcal{M}} (q_h, L\underline{b}R)$$

for some $b \in B$ and some tape instances L and R. A set $S \subseteq B^\star$ is ***recursively enumerable (r.e.,)*** *or* ***semi-decidable****, respectively* ***co-recursively enumerable (co-r.e.)*** *or* ***co-semi-decidable****, iff there is some Turing machine $\mathcal{M}$ which terminates on precisely all inputs $b_1b_2\cdots b_n \in S$, respectively on precisely all inputs $b_1b_2\cdots b_n \notin S$, and is* ***recursive*** *or* ***decidable*** *iff it is both r.e. and co-r.e.*

Note that a Turing machine as we defined it cannot get stuck in any state but q_h, because the mapping M is defined everywhere except in q_h. Therefore, for any given input, a Turing machine carries out a determined succession of steps, which may or may not terminate. A Turing machine can be regarded as an idealized, low-level programming language, which can be used for computations by placing a certain input on the tape and letting it run; if it terminates, the result of the computation can be found on the tape. Since our Turing machines have only symbols 0 and 1, one has to use them ingeniously to encode more complex inputs, such as natural numbers. There are many different ways to do this. A simple tape representation of natural numbers is to represent a number n by n consecutive cells containing the bit 1. This works, however, only when n is strictly larger than 0. Another representation, which also accommodates $n = 0$, is as a sequence of $n + 1$ cells containing 1. With this latter representation, one can then define Turing machines corresponding

States and transition function (graphical representation to the right):

$Q = \{q_s, q_h, q_1, q_2\}$
$M(q_s, 0) = (q_1, \rightarrow)$
$M(q_s, 1) = \text{anything}$
$M(q_1, 0) = (q_2, 1)$
$M(q_1, 1) = (q_1, \rightarrow)$
$M(q_2, 0) = (q_h, 0)$
$M(q_2, 1) = (q_2, \leftarrow)$



Sample computation:

$(q_s, 0^\omega \underline{0}1110^\omega) \rightarrow_{\mathcal{M}} (q_1, 0^\omega \underline{1}110^\omega) \rightarrow_{\mathcal{M}} (q_1, 0^\omega 1\underline{1}10^\omega) \rightarrow_{\mathcal{M}} (q_1, 0^\omega 11\underline{1}0^\omega) \rightarrow_{\mathcal{M}}$
$(q_1, 0^\omega 111\underline{0}0^\omega) \rightarrow_{\mathcal{M}} (q_2, 0^\omega 111\underline{1}0^\omega) \rightarrow_{\mathcal{M}} (q_2, 0^\omega 11\underline{1}10^\omega) \rightarrow_{\mathcal{M}} (q_2, 0^\omega 1\underline{1}110^\omega) \rightarrow_{\mathcal{M}}$
$(q_2, 0^\omega \underline{1}1110^\omega) \rightarrow_{\mathcal{M}} (q_2, 0^\omega \underline{0}11110^\omega) \rightarrow_{\mathcal{M}} (q_h, 0^\omega \underline{0}11110^\omega)$

Figure 2.1: Turing machine $\mathcal{M}$ computing the successor function, and sample computation

to functions that take any natural numbers as input and produce any natural numbers as output. For example, Figure 2.1 shows a Turing machine computing the successor function. Cells containing 0 can then be used as number separators, when more natural numbers are needed. For example, a Turing machine computing a binary operation on natural numbers would run on configurations $(q_s, 0^\omega \underline{0}1^{m+1}01^{n+1}0^\omega)$ and would halt on configurations $(q_h, 0^\omega \underline{0}1^{k+1}0^\omega)$, where m, n, k are natural numbers.

One can similarly have tape encodings of rational numbers; for example, one can encode the number m/n as m followed by n with two 0 cells in-between (and keep the one-0-cell-convention for argument separation). Real numbers are not obviously representable, though. A Turing machine is said to *compute* a real number r iff it can finitely approximate r (for example using a rational number) with any desired precision; one way to formalize this is as follows: Turing machine $\mathcal{M}_r$ computes r iff when run on input natural number p, it halts with result rational number m/n such that $|r - m/n| < 1/10^p$. If a real number can be computed by a Turing machine then it is called *Turing computable*. Many real numbers, e.g., π, e, $\sqrt{2}$, etc., are Turing computable.

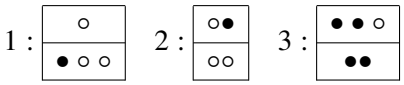
2.2.2 Universal Machines, the Halting Problem, and Decision Problems

Since Turing machines have finite descriptions, they can be encoded themselves as natural numbers. Therefore, we can refer to "the k^{th} Turing machine", where k is a natural number, the same way we can refer to the i^{th} input to a Turing machine. A *universal Turing machine* is a Turing machine that can simulate an arbitrary Turing machine on arbitrary input. The universal machine essentially achieves this by reading both the description of the machine to be simulated as well as the input thereof from its own tape. There are various constructions of universal Turing machines in the literature, which we do not repeat here. We only notice that we can construct such a universal machine $\mathcal{U}$ which terminates precisely on all inputs of the form 1^k01^i where Turing machine k terminates on input i. This immediately implies that the language $\{1^k01^i \mid \text{Turing machine } k \text{ terminates on input } i\}$ is recursively enumerable. However, the undecidability of the famous *halting problem* (does a given Turing machine terminates on a given input?) implies that this language is not recursive; more specifically, it is not co-recursively enumerable.

Since the elements of many mathematical domains can be encoded as words in $B^\star$, the terminology in Definition 8 is also used for *decision problems* over such domains. For example, the decision problem of

whether a graph given as input has a cycle or not is recursive; in other words, the set of cyclic graphs (under some appropriate encoding in $B^\star$) is recursive/decidable. Since there is a bijective correspondence between elements in $B^\star$ and natural numbers, and also between tuples of natural numbers and natural numbers, decision problems are often regarded as one- or multi-argument relations or predicates over natural numbers. For example, a subset $R \subseteq B^\star$ can be regarded as a predicate, say of three natural number arguments, where $R(i, j, k)$ for $i, j, k \in Nat$ indicates that the encoding of (i, j, k) belongs to R.

While the haling problem is typically an excellent vehicle to formally state and prove that certain problems are undecidable, so they cannot be solved by computers no matter how powerful they are or what programming languages are used, its reflective nature makes the halting problem sometimes hard to use in practice. The *Post correspondence problem (PCP)* is an another canonical undecidable problem, which is sometimes easier to use to show that other problems are undecidable. The PCP can be stated as follows: given a set of (domino-style) tiles each containing a top and a bottom string of 0/1 bits, is it possible to find a sequence of possibly repeating such tiles so that the concatenated top strings equal the concatenated bottom strings? For example, if the given tiles are the following:



then the answer to the PCP problem is positive, because the sequence of tiles 3 2 3 1 yields the same concatenated strings at the top and at the bottom.

2.2.3 The Arithmetic Hierarchy

The *arithmetical hierarchy* defines classes of problems of increasing difficulty, called *degrees*, as follows:

$$\Sigma^0_0 = \Pi^0_0 = \{R \mid R \text{ recursive}\}$$

$$\Sigma^0_{n+1} = \{P \mid \exists Q \in \Pi^0_n,\ \forall i(P(i) \leftrightarrow \exists j Q(i, j))\}$$

$$\Pi^0_{n+1} = \{P \mid \exists Q \in \Sigma^0_n,\ \forall i(P(i) \leftrightarrow \forall j Q(i, j))\}$$

For example, the Σ^0_1 degree consists of the predicates P over natural numbers for which there is some recursive predicate R such that for any $i \in Nat$, $P(i)$ holds iff $R(i, j)$ holds for some $j \in Nat$. It can be shown that Σ^0_1 contains precisely the recursively enumerable predicates. Similarly, Π^0_1 consists of the predicates P for which there is some recursive predicate R such that for any $i \in Nat$, $P(i)$ holds iff $R(i, j)$ holds for all $j \in Nat$, which is precisely the set of co-recursively enumerable predicates. An important degree is also Π^0_2, which consists of the predicates P over natural numbers for which there is some recursive predicate R such that for any $i \in Nat$, $P(i)$ holds iff for any $j \in Nat$ there is some $k \in Nat$ such that $R(i, j, k)$ holds. A prototypical Π^0_2 problem is the following: giving a Turing machine $\mathcal{M}$, does it terminate on all inputs? The complexity of the problem stays in the fact that there are infinitely many (but enumerable) inputs, so one can never be "done" with testing them; moreover, even for a given input, one does not know when to stop running the machine and reject the input. However, if one is given for each input accepted by the machine a run of the machine, then one can simply check the run and declare the input indeed accepted. Therefore, $\mathcal{M}$ terminates on all inputs iff for any input there exists some accepting run of $\mathcal{M}$, which makes it a Π^0_2 problem because checking whether a given run on a given Turing machine with a given input accepts the input is decidable. Moreover, it can be shown that if we pick $\mathcal{M}$ to be the universal Turing machine $\mathcal{U}$ discussed above, then the following problem, which we refer to as Totality from here on, is in fact Π^0_2-complete:

INPUT: A natural number k;
OUTPUT: Does $\mathcal{U}$ terminate on all inputs 1^k01^i for all $i \geq 0$?

For any n, both degrees Σ_n^0 and Π_n^0 are properly included in both the immediately above degrees Σ_{n+1}^0 and Π_{n+1}^0. There are extensions which define degrees Σ_n^1, Π_n^2, etc., but we do not discuss them here.

2.2.4 Notes

The abstract computational model that we call the "Turing machine" today was originally called "the computer" when proposed by Alan Turing in 1936-37 [80]. In the original article, Turing imagined not a machine, but a person (the computer) who executes a series of deterministic mechanical rules "in a desultory manner". In his 1948 essay "Intelligent Machinery", Turing calls the machine he proposed the *logical computing machine*, a name which has not been adopted, everybody preferring to call it the *Turing machine*.

It is insightful to understand the scientific context in which Turing proposed his machine. In the 1930s, there were several approaches attempting to address Hilbert's tenth question of 1900, the *Entscheidungsproblem* (the *decision problem*). Partial answers have been given by Kurt Gödel in 1930 (and published in 1931), under the form of his famous *incompleteness theorems*, and in 1934, under the form of his *recursion theory*. Alonzo Church is given the credit for being the first to effectively show that the Entscheidungsproblem was indeed *undecidable*, introducing also λ-calculus (discussed in Section 4.5). Church published his paper on 15 April 1936, about one month before Turing submitted his paper on 28 May 1936. In his paper, Turing also proved the equivalence of his machine to Church's λ-calculus. Interestingly, Turing's paper was submitted only a few months before Emil Post, another great logician, submitted a paper independently proposing an almost identical computational model on 7 October 1936 [62]. The major difference between Turing's and Post's machines is that the former uses a tape which is infinite in only one direction, while the latter works with a tape which is infinite at both ends, like the "Turing machine" that we used in this section. We actually took our definitions in this section from Rogers' book [67], which we recommend the reader for more details in addition to comprehensive textbooks such as Sipser [76] and Hopcroft, Motwani and Ullman [33]. To remember the fact that Post and Turing independently invented an almost identical computational model of utmost importance, several computer scientists call it the "Post-Turing machine".

Even though Turing was not the first to propose what we call today a Turing-complete model of computation, many believe that his result was stronger than Church's, in that his computational model was more direct, easier to understand, and based on first, low-level computational principles. For example, it is typically easy to implement Turing Machines in any programming language, which is not necessarily the case for other Turing-complete models, such as, for example, the λ-calculus. As seen in Section 4.5, λ-calculus relies on a non-trivial notion of substitution, which comes with the infamous *variable capture* problem.

2.2.5 Exercises

Exercise 12. *Define Turing machines corresponding to the addition, multiplication, and power operations on natural numbers. For example, the initial configuration of the Turing machine computing addition with 3 and 7 as input is* $(q_s, 0^\omega\underline{0}111101111111110^\omega)$*, and its final configuration is* $(q_h, 0^\omega\underline{0}1111111111110^\omega)$*. We here assumed that n is encoded as* $n + 1$ *cells containing 1.*

Exercise 13. *Show that there are real numbers which are not Turing computable.*
(Hint: The set of Turing machines is recursively enumerable.)

sorts:
Cell, *Tape*, *Configuration*
operations:
$0, 1 : \rightarrow Cell$
$zeros : \rightarrow Tape$
$\_ : \_ : Cell \times Tape \rightarrow Tape$
$q : Tape \times Tape \rightarrow Configuration$ — one such operation for each $q \in Q$
generic equation:
$zeros = 0 : zeros$
specific equations:
$q(L, b:R) = q'(L, b':R)$ — one equation for each $q, q' \in Q$, $b, b' \in Cell$ with $M(q, b) = (q', b')$
$q(L, b:R) = q'(b:L, R)$ — one equation for each $q, q' \in Q$, $b \in Cell$ with $M(q, b) = (q', \rightarrow)$
$q(B:L, b:R) = q'(L, B:b:R)$ — one equation for each $q, q' \in Q$, $b \in Cell$ with $M(q, b) = (q', \leftarrow)$

Figure 2.2: Lazy equational logic representation $\mathcal{E}_{\mathcal{M}}^{lazy}$ of Turing machine $\mathcal{M}$

2.4.3 Computation as Equational Deduction

Here we discuss simple equational logic encodings of Turing machines (see Section 2.2.1 for general Turing machine notions). The idea is to associate an equational theory to any Turing machine, so that an input is accepted by the Turing machine if and only if an equation corresponding to that input can be proved from the equational theory of the Turing machine, using conventional equational deduction. Moreover, as seen in Section 2.5.3, the resulting equational theories can be executed as rewrite theories by rewrite engines, thus yielding actual Turing machine interpreters. We present two encodings, both based on intuitions from lazy data-structures, specifically stream data-structures. The first is simpler but requires lazy rewriting support from rewrite engines in order to be executed, while the second can be executed by any rewrite engines.

Lazy Equational Representation

Our first representation of Turing machines in equational logic is based on the idea that the infinite tape can be finitely represented by means of self-expanding stream data-structures. In spite of being infinite sequences of cells, like the Turing machine tapes, many interesting streams can be finitely specified using equations. For example, the stream of zeros, $zeros = 0 : 0 : 0 : \cdots$, can be defined as $zeros = 0 : zeros$. Since at any given moment the portions of a Turing machine tape to the left and to the right of the head have a suffix consisting of an infinite sequence of 0 cells, it is natural to represent them as streams of the form $b_1{:}b_2{:}\cdots b_n{:}zeros$. When the head is on cell b_n and the command is to move the head to the right, the self-expanding equational definition of *zeros* can produce one more 0, so that the head can move onto it. To expand *zeros* on a by-need basis and thus to avoid undesired non-termination due to the uncontrolled application of the self-expanding equation of *zeros*, this approach requires an equational/rewrite engine with support for lazy evaluation/rewriting in order to be executed.

Figure 2.2 shows how a Turing machine $\mathcal{M} = (Q, B, q_s, q_h, C, M)$ can be associated a computationally equivalent equational logic theory $\mathcal{E}_{\mathcal{M}}^{lazy}$. Except for the self-expanding equation of the *zeros* stream and our stream representation of the two-infinite-end tape, the equations of $\mathcal{E}_{\mathcal{M}}^{lazy}$ are identical to the transition relation on Turing machine configurations discussed right after Definition 7. The self-expanding equation of *zeros*

guarantees that enough 0's can be provided when the head reaches one or the other end of the sequence of cells visited so far. The result below shows that $\mathcal{E}_{\mathcal{M}}^{lazy}$ is proof-theoretically equivalent to $\mathcal{M}$:

Theorem 7. *The following are equivalent:*

(1) The Turing machine $\mathcal{M}$ terminates on input $b_1 b_2 \dots b_n$;

(2) $\mathcal{E}_{\mathcal{M}}^{lazy} \models q_s(zeros, b_1 : b_2 : \cdots : b_n : zeros) = q_h(l, r)$ for some terms l, r of sort Tape.

Proof. ... □

The equations in $\mathcal{E}_{\mathcal{M}}^{lazy}$ can be applied in any direction, so an equational proof of $\mathcal{E}_{\mathcal{M}}^{lazy} \models q_s(zeros, b_1 : b_2 : \cdots : b_n : zeros) = q_h(l, r)$ needs not necessarily correspond step-for-step to the computation of $\mathcal{M}$ on input $b_1 b_2 \dots b_n$. We will see in Section 2.5.3 that by orienting the specific equations in Figure 2.2 into rewrite rules, we will obtain a rewrite logic theory which will faithfully capture, step-for-step, the computational granularity of $\mathcal{M}$.

Note that in Figure 2.2 we preferred to define a configuration construct $q : Tape \times Tape \to Configuration$ for each $q \in Q$. A natural alternative could have been to define an additional sort *State* for the Turing machine states, a constant $q : \to State$ for each $q \in Q$, and one generic configuration construct $\_(\_, \_) : State \times Tape \times Tape \to Configuration$, as we do in the subsequent representation of Turing machines as rewriting logic theories (see Figure 2.3). The reason for which we did not do that here is twofold: first, in functional languages like Haskell it is very natural to associate a function to each such configuration construct $q : Tape \times Tape \to Configuration$, while it would take some additional effort to implement the second approach; second, the approach in this section is more compact than the one below.

Unrestricted Equational Representation

The equational representation of Turing machines above is almost as simple as it can be and, additionally, can be easily executed on programming languages or rewrite engines with support for lazy evaluation/rewriting, such as Haskell or Maude (see, e.g., Section 2.5.6). However, the fact that it requires lazy evaluation/rewriting and that the equivalence classes of configurations have infinitely many terms, its use is limited to systems that support strategies. Here we show that a simple idea can turn the representation in the previous section into an elementary one which can be executed on any equational/rewrite engines: replace the self-expanding and non-terminating (when regarded as a rewrite rule) equation "$zeros = 0{:}zeros$" with configuration equations of the form "$q(zeros, R) = q(0{:}zeros, R)$" and "$q(L, zeros) = q(L, 0{:}zeros)$"; these equations achieve the same role of expanding *zeros* by need, but avoid non-termination when applied as rewrite rules.

Figure 2.3 shows our unrestricted representation of Turing machines as equational logic theories. There are some minor differences between the representation in Figure 2.3 and the one in Figure 2.2. For example, note that in order to add the two equations above for the expanding of *zeros* in a generic manner for any state, we separate the states from the configuration construct. In other words, instead of having an operation $q : Tape \times Tape \to Configuration$ for each $q \in Q$ like in Figure 2.2, we now have one additional sort *State*, a generic configuration construct $\_(\_, \_) : State \times Tape \times Tape \to Configuration$, and a constant $q : \to State$ for each $q \in Q$. This change still allows us to write configurations as terms $q(l, r)$, so we do not need to change the equations corresponding to the Turing machine transitions. With this modification in the signature, we can now remove the troubling equation $zeros = 0{:}zeros$ from the representation in Figure 2.2 and replace it with the two safe equations in Figure 2.3. Let $\mathcal{E}_{\mathcal{M}}$ be the equational logic theory in Figure 2.3.

Theorem 8. *The following are equivalent:*

sorts:
Cell, *Tape*, *State*, *Configuration*
operations:
$0, 1 \ : \ \rightarrow \mathit{Cell}$
$\mathit{zeros} \ : \rightarrow \mathit{Tape}$
$\_ : \_ \ : \ \mathit{Cell} \times \mathit{Tape} \rightarrow \mathit{Tape}$
$\_(\_, \_) \ : \ \mathit{State} \times \mathit{Tape} \times \mathit{Tape} \rightarrow \mathit{Configuration}$
$q \ : \rightarrow \mathit{State}$ — one such constant for each $q \in Q$
generic equations:
$S(\mathit{zeros}, R) = S(0{:}\mathit{zeros}, R)$
$S(L, \mathit{zeros}) = S(L, 0{:}\mathit{zeros})$
specific equations:
$q(L, b{:}R) = q'(L, b'{:}R)$ — one equation for each $q, q' \in Q$, $b, b' \in \mathit{Cell}$ with $M(q, b) = (q', b')$
$q(L, b{:}R) = q'(b{:}L, R)$ — one equation for each $q, q' \in Q$, $b \in \mathit{Cell}$ with $M(q, b) = (q', \rightarrow)$
$q(B{:}L, b{:}R) = q'(L, B : b{:}R)$ — one equations for each $q, q' \in Q$, $b \in \mathit{Cell}$ with $M(q, b) = (q', \leftarrow)$

Figure 2.3: Unrestricted equational logic representation $\mathcal{E}_{\mathcal{M}}$ of Turing machine $\mathcal{M}$

(1) The Turing machine $\mathcal{M}$ terminates on input $b_1 b_2 \ldots b_n$;

(2) $\mathcal{E}_{\mathcal{M}} \models q_s(\mathit{zeros}, b_1{:}\, b_2{:} \cdots {:}\, b_n{:}\mathit{zeros}) = q_h(l, r)$ for some terms l, r of sort Tape.

Proof. ... □

One could argue that deduction with the equational theory in Figure 2.3 is not fully faithful to computations with the original Turing machine, because the two generic equations may need to artificially apply from time to time as an artifact of our representation, and their application does not correspond to actual computational steps in the Turing machine. In fact, these generic equations can be completely eliminated, at the expense of more equations. For example, if $M(q, b) = (q', \leftarrow)$ then, in addition to the last equation in Figure 2.3, we can also include the equation:

$$q(\mathit{zeros}, b{:}R) = q'(\mathit{zeros}, 0 : b{:}R)$$

This way, one can expand *zeros* and apply the transition in *one* equational step. Doing that systematically for all the transitions allows us to eliminate the need for the two generic equations entirely.

sort:

Stream

operations:

$$
\begin{array}{rcl}
\_ : \_ & : & \mathit{Int} \times \mathit{Stream} \to \mathit{Stream} \\
\mathit{head} & : & \mathit{Stream} \to \mathit{Int} \\
\mathit{tail} & : & \mathit{Stream} \to \mathit{Stream} \\
\mathit{zeros} & : & \to \mathit{Stream} \\
\mathit{zip} & : & \mathit{Stream} \times \mathit{Stream} \to \mathit{Stream} \\
\mathit{add} & : & \mathit{Stream} \to \mathit{Stream} \\
\mathit{fibonacci} & : & \to \mathit{Stream}
\end{array}
$$

equations:

$$
\begin{array}{l}
\mathit{head}(X : S) = X \\
\mathit{tail}(X : S) = S \\
\mathit{zeros} = 0 : \mathit{zeros} \\
\mathit{zip}(X : S_1, S_2) = X : \mathit{zip}(S_2, S_1) \\
\mathit{add}(X_1 : X_2 : S) = (X_1 +_{\mathit{Int}} X_2) : \mathit{add}(S) \\
\mathit{fibonacci} = 0 : 1 : \mathit{add}(\mathit{zip}(\mathit{fibonacci}, \mathit{tail}(\mathit{fibonacci})))
\end{array}
$$

Figure 2.8: Streams of integers defined as an algebraic datatype. The variables S, S_1, S_2 have sort *Stream* and the variables X, X_1, X_2 have sort *Int*.

Streams

Figure 2.8 shows an example of a data-structure whose elements are infinite sequences, called *streams*, together with several particular streams and operations on them. Here we prefer to be more specific than in the previous examples and work with streams of integers. We assume the integers and operations on them already defined; specifically, we assume *Int* to be their sort and operations on them indexed with *Int* to distinguish them from other homonymous operations, e.g., $+_{\mathit{Int}}$, etc. The operation $\_ : \_$ adds a given integer to the beginning of a given stream, and the dual operations *head* and *tail* extract the head (integer) and the tail (stream) from a stream. The stream *zeros* contains only 0 elements. The stream operation *zip* merges two streams by interleaving their elements, and *add* generates a new stream by adding any two consecutive elements of a given stream. The stream *fibonacci* consists of the Fibonacci sequence (see Exercise 25).

It is interesting to note that the equational specification of streams in Figure 2.8 is one where its initial algebra semantics is likely *not* the model that we want. Indeed, the initial algebra here would consists of infinite classes of finite terms, where any two terms in any class are provably equal, for example $\{\mathit{zeros},\ 0 : \mathit{zeros},\ 0 : 0 : \mathit{zeros}, \dots\}$. While this is a valid and interesting model of streams, it is likely not what one has in mind when one thinks of streams as infinite sequences. Nevertheless, the intended stream model is among the models/algebras of this equational specification, so any equational deduction or reduction that we perform, with or without strategies, is sound (see Exercise 26).

2.4.7 Notes

Equational encodings of general computation into equational deduction are well-known; for example, [7, 1] show such encodings, where the resulting equational specifications, if regarded as term rewrite systems (TRSs), are confluent and terminate whenever the original computation terminates. Our goal in this section is to discuss equational encodings of (Turing machine) computation. These encodings will be used later in the paper to show the Π^0_2-hardness of the equational satisfaction problem in the initial algebra. While we could have used existing encodings of Turing machines as TRSs, however, we found them more complex and intricate for our purpose in this paper than needed. Consequently (and also for the sake of self-containment), we recall the more recent (simple) encoding and corresponding proofs from [65]. Since the subsequent encoding is general purpose rather than specific to our Π^0_2-hardness result, the content of this section may have a more pedagogical than technical nature. For example, the references to TRSs are technically only needed to prove the equational encoding correct, so they could have been removed from the main text and added only in the proofs, but we find them pedagogically interesting and potentially useful for other purposes. The equational encodings that follow can be faithfully used as TRS Turing-complete computational engines, because each rewrite step corresponds to precisely one computation step in the Turing machine; in other words, there are no artificial rewrite steps.

2.4.8 Exercises

Exercise 24. *Eliminate the two equations in Figure 2.3 as discussed right after Theorem 8, and prove a result similar to Theorem 8 for the new representation.*

Exercise 25. *Show that the fibonacci stream defined in Figure 2.8 indeed defines the sequence of Fibonacci numbers. This exercise has two parts: first formally state what to prove, and second prove it.*

Exercise 26. *Consider the equational specification of streams in Figure 2.8. Define the intended model/algebra of streams over integer numbers with constant streams and functions on streams corresponding to the various operations in this specification. Then show that this model indeed satisfies all the equations in Figure 2.8. Describe also its default initial model and compare it with the intended model. Are they isomorphic?*

sorts:
$Cell$, $Tape$, $Configuration$
operations:
$0, 1 : \rightarrow Cell$
$zeros : \rightarrow Tape$
$\_ : \_ : Cell \times Tape \rightarrow Tape$
$q : Tape \times Tape \rightarrow Configuration$ — one such operation for each $q \in Q$
equations:
$zeros = 0 : zeros$
rules:
$q(L, b:R) \rightarrow q'(L, b':R)$ — one rule for each $q, q' \in Q$, $b, b' \in Cell$ with $(q', b') \in M(q, b)$
$q(L, b:R) \rightarrow q'(b:L, R)$ — one rule for each $q, q' \in Q$, $b \in Cell$ with $(q', \rightarrow) \in M(q, b)$
$q(B:L, b:R) \rightarrow q'(L, B:b:R)$ — one rule for each $q, q' \in Q$, $b \in Cell$ with $(q', \leftarrow) \in M(q, b)$

Figure 2.10: Lazy rewriting logic representation $\mathcal{R}_{\mathcal{M}}^{lazy}$ of Turing machine $\mathcal{M}$

2.5.3 Computation as Rewriting Logic Deduction

Building upon the equational representations of deterministic Turing machines in Section 2.4.3, here we show how we can associate rewrite theories to non-deterministic Turing machines so that there is a bijective correspondence between computational steps performed by the original Turing machine and rewrite steps in the corresponding rewrite theory. In non-deterministic Turing machines, the total transition function $M : (Q - \{q_h\}) \times B \rightarrow Q \times C$ generalizes to a total relation, or in other words to a function into the strict powerset of $Q \times C$, $M : (Q - \{q_h\}) \times B \rightarrow \mathcal{P}^+(Q \times C)$, that is, taking each non-halting state q and current cell bit contents b into a non-empty set $M(q, b)$ of non-deterministic (state,action) choices. For example, to turn the successor Turing machine in Figure 2.1 into one which non-deterministically chooses to add one more 1 to the given number *or not* when it reaches its end, all we have to do is to modify its transition function in state q_1 and cell contents 0 to return two possible continuations: $M(q_1, 0) = \{(q_2, 1), (q_2, \leftarrow)\}$. Like in Section 2.4.3, we give both lazy and unrestricted representations.

Lazy Rewrite Logic Representation

Figure 2.10 shows how a Turing machine $\mathcal{M}$ can be associated a computationally equivalent rewriting logic theory $\mathcal{R}_{\mathcal{M}}^{lazy}$. The only difference between this rewrite logic theory and the equational logic theory in Figure 2.2 is that the equations which were specific to the Turing machine have been turned into rewrite rules. The equation expanding the stream of zeros remains an equation. Since in rewriting logic only the rewrite rules count as transitions, and they apply modulo equations, the rewrite theory is in fact more faithful to the actual computational steps embodied in the Turing machine. The result below formalizes this by showing that there is a step-for-step equivalence between computations using $\mathcal{M}$ and rewrites using $\mathcal{R}_{\mathcal{M}}^{lazy}$:

Theorem 10. *The rewriting logic theory $\mathcal{R}_{\mathcal{M}}^{lazy}$ is confluent. Moreover, the Turing machine $\mathcal{M}$ and the rewrite theory $\mathcal{R}_{\mathcal{M}}^{lazy}$ are step-for-step equivalent, that is,*

$$(q, 0^\omega \underline{u} \underline{b} v 0^\omega) \rightarrow_{\mathcal{M}} (q', 0^\omega u' \underline{b'} \underline{v'} 0^\omega) \quad \textit{if and only if} \quad \mathcal{R}_{\mathcal{M}}^{lazy} \models q(\overleftarrow{u}, b : \overrightarrow{v}) \rightarrow^1 q'(\overleftarrow{u'}, b' : \overrightarrow{v'})$$

sorts:
$Cell$, $Tape$, $State$, $Configuration$
operations:
$0, 1 : \rightarrow Cell$
$zeros : \rightarrow Tape$
$\_:\_ : Cell \times Tape \rightarrow Tape$
$\_(\_,\_) : State \times Tape \times Tape \rightarrow Configuration$
$q : \rightarrow State$ — one such constant for each $q \in Q$
equations:
$S(zeros, R) = S(0{:}zeros, R)$
$S(L, zeros) = S(L, 0{:}zeros)$
rules:
$q(L, b{:}R) \rightarrow q'(L, b'{:}R)$ — one rule for each $q, q' \in Q$, $b, b' \in Cell$ with $(q', b') \in M(q, b)$
$q(L, b{:}R) \rightarrow q'(b{:}L, R)$ — one rule for each $q, q' \in Q$, $b \in Cell$ with $(q', \rightarrow) \in M(q, b)$
$q(B{:}L, b{:}R) \rightarrow q'(L, B : b{:}R)$ — one rule for each $q, q' \in Q$, $b \in Cell$ with $(q', \leftarrow) \in M(q, b)$

Figure 2.11: Unrestricted rewriting logic representation $\mathcal{R}_{\mathcal{M}}$ of Turing machine $\mathcal{M}$

for any finite sequences of bits $u, v, u', v' \in \{0,1\}^*$, any bits $b, b' \in \{0,1\}$, and any states $q, q' \in Q$, where if $u = b_1 b_2 \ldots b_{n-1} b_n$, then $\overleftarrow{u} = b_n : b_{n-1} : \cdots : b_2 : b_1 : zeros$ and $\overrightarrow{u} = b_1 : b_2 : \cdots : b_{n-1} : b_n : zeros$. Finally, the following are equivalent:

(1) The Turing machine $\mathcal{M}$ terminates on input $b_1 b_2 \ldots b_n$;

(2) $\mathcal{R}_{\mathcal{M}}^{lazy} \models q_s(zeros, b_1{:}\, b_2{:} \cdots {:}\, b_n{:}zeros) \rightarrow q_h(l, r)$ for some terms l, r of sort Tape; note though that $\mathcal{R}_{\mathcal{M}}^{lazy}$ does not terminate on term $q_s(zeros, b_1{:}\, b_2{:} \cdots {:}\, b_n{:}zeros)$ as an unrestricted rewrite system, since the equation $zeros = 0 : zeros$ (regarded as a rewrite rule) can apply forever, thus yielding infinite equational classes of configurations with no canonical forms, but $\mathcal{R}_{\mathcal{M}}^{lazy}$ terminates on $q_s(zeros, b_1{:}\, b_2{:} \cdots {:}\, b_n{:}zeros)$ if the stream construct operation $\_:\_ : Cell \times Tape \rightarrow Tape$ has a lazy rewriting strategy on its second argument;

Proof. ... □

Therefore, unlike the equational logic theory $\mathcal{E}_{\mathcal{M}}^{lazy}$ in Theorem 7, the rewrite logic theory $\mathcal{R}_{\mathcal{M}}^{lazy}$ faithfully captures, step-for-step, the computational granularity of $\mathcal{M}$. Recall that equational deduction does not count as computational, or rewrite steps in rewriting logic, which allows to apply the self-expanding equation of *zeros* silently in the background. Since there are no artificial rewrite steps, we can conclude that $\mathcal{R}_{\mathcal{M}}$ actually *is* precisely $\mathcal{M}$ and not an encoding of it. Theorem 10 thus showed not only that rewriting logic is Turing complete, but also that it faithfully captures the computational granularity of the represented Turing machines.

Unrestricted Rewrite Logic Representations

Figure 2.11 shows our unrestricted representation of Turing machines as rewriting logic theories, following the same idea as the equational representation in Section 2.4.3 (Figure 2.3). Let $\mathcal{R}_{\mathcal{M}}$ be the rewriting logic theory in Figure 2.11. Then the following result holds:

Theorem 11. *The rewriting logic theory $\mathcal{R}_\mathcal{M}$ is confluent. Moreover, the Turing machine $\mathcal{M}$ and the rewrite theory $\mathcal{R}_\mathcal{M}$ are step-for-step equivalent, that is,*

$$(q, 0^\omega u\underline{b}v0^\omega) \to_\mathcal{M} (q', 0^\omega u'\underline{b'}v'0^\omega) \quad \textit{if and only if} \quad \mathcal{R}_\mathcal{M} \models q(\overleftarrow{u}, b : \overrightarrow{v}) \to^1 q'(\overleftarrow{u'}, b' : \overrightarrow{v'})$$

for any finite sequences of bits $u, v, u', v' \in \{0,1\}^*$, any bits $b, b' \in \{0,1\}$, and any states $q, q' \in Q$, where if $u = b_1 b_2 \ldots b_{n-1} b_n$, then $\overleftarrow{u} = b_n : b_{n-1} : \cdots : b_2 : b_1 : zeros$ and $\overrightarrow{u} = b_1 : b_2 : \cdots : b_{n-1} : b_n : zeros$. Finally, the following are equivalent:

(1) The Turing machine $\mathcal{M}$ terminates on input $b_1 b_2 \ldots b_n$;

(2) $\mathcal{R}_\mathcal{M}$ terminates on term $q_s(zeros, b_1 : b_2 : \cdots : b_n : zeros)$ as an unrestricted rewrite system and $\mathcal{R}_\mathcal{M} \models q_s(zeros, b_1 : b_2 : \cdots : b_n : zeros) \to q_h(l, r)$ for some terms l, r of sort Tape;

Proof. ... □

Like for the lazy representation of Turing machines in rewriting logic discussed above, the rewrite theory $\mathcal{R}_\mathcal{M}$ *is* the Turing machine $\mathcal{M}$, in that there is a step-for-step equivalence between computational steps in $\mathcal{M}$ and rewrite steps in $\mathcal{R}_\mathcal{M}$. Recall, again, that equations do not count as rewrite steps, their role being to structurally rearrange the term so that rewrite rules can apply; indeed, that is precisely the intended role of the two equations in Figure 2.11 (they reveal new blank cells on the tape whenever needed). Similarly to the equational case in Section 2.4.3, the two generic equations can be completely eliminated. However, this time we have to add more Turing-machine-specific rules instead. For example, if $(q', \leftarrow) \in M(q, b)$ then, in addition to the last rule in Figure 2.11, we also include the rule:

$$q(zeros, b : R) \to q'(zeros, 0 : b : R)$$

This way, one can expand *zeros* and apply the transition in *one* rewrite step, instead of one equational step and one rewrite step. Doing that systematically for all the transitions allows us to eliminate the need for equations entirely; the price to pay is, of course, that the number of rules increases.

2.5.6 Maude: A High Performance Rewrite Logic System

Maude (`http://maude.cs.uiuc.edu`) is a rewrite logic *executable specification language*, which builds upon a fast rewrite engine. Our main use of Maude in this book is as a platform to execute rewrite logic semantic definitions, following the various approaches in Chapter 3. Our goal here is to give a high-level overview of Maude, mentioning only its features needed in this book. We make no attempt to give a systematic presentation of Maude here, meant to replace its manual or other more comprehensive papers or books (some mentioned in Section 2.5.8). The features we need will be introduced on-the-fly, with enough explanations to make this book self-contained, but the reader interested in learning Maude in depth should consult its manual.

We will use Maude to specify programming language features, which, when put together via simple Maude module operations, lead to programming language semantic definitions. Since Maude is executable, *interpreters for programming languages designed this way will be obtained for free*, which will be very useful for understanding, refining and/or changing the languages. Moreover, *formal analysis tools* for the specified languages can also be obtained with little effort, such as exhaustive state-space searchers or model-checkers, simply by using the corresponding generic rewrite logic analysis tools already provided by Maude.

Devising *formal semantic executable models* of desired languages or tools *before these are implemented* is a crucial step towards a deep understanding of the language or tool. In simplistic terms, it is like devising a simulator for an expensive system before building the actual system. However, our simulators in this book will consist of exactly the *semantics*, or the *meaning*, of our desired systems, defined using a very rigorous, mathematical notation. In the obtained formal executable model of a programming language, executing a program will correspond to nothing but *logical inference* within the semantics of the language.

How to Execute Maude

After installing Maude on your platform and setting up the environment path variable, you should be able to type `maude` and immediately see a welcome screen followed by a cursor waiting for user input:

```
Maude>
```

Maude is interpreted, so you can just type your specifications and commands. However, a more practical way is to type everything in one file, say `pgm.maude`, and then include that file with the command

```
Maude> in pgm.maude
```

after starting Maude (the extension is optional), or, alternatively, start Maude with `pgm` as an argument: "`maude pgm`". In both cases, the contents of `pgm.maude` will be loaded and executed as if it was manually typed at the cursor. Use the `quit` command, or simply `q`, to quit Maude. Since Maude's initialization and termination are quite fast, many users end their `pgm.maude` file with a `q` command on a new line, so that Maude terminates as soon as the program is executed. Another useful command in files is `eof`, which tells Maude that the end-of-file is meant there and thus it does not process the code following the `eof` command. Instead, the control is given to the user, who can manually type commands, etc. You can correct/edit your Maude definition in `pgm.maude` and then load it again. However, keep it in mind that Maude maintains only one working session, in particular one module database, until you quit it. This can sometimes lead to unexpected errors for beginners, so if you are not sure about an error just quit and then restart Maude.

Modules

Maude specifications are introduced as *modules*. There are several kinds of modules, but for simplicity we only use general modules in this book, which have the syntax

```
mod <NAME> is
  <BODY>
endm
```

where <NAME> can be any identifier. The <BODY> of a module can include importation of other modules, sort and operation declarations, and a set of sentences. The sorts together with the operations form the *signature* of that module, and can be thought of as the interface of that module to other modules.

To lay the ground for introducing more Maude features, let us define Peano-style natural numbers with addition and multiplication. We define the addition first, in one separate module:

```
mod PEANO-NAT is
  sort Nat .
  op zero : -> Nat .
  op succ : Nat -> Nat .
  op plus : Nat Nat -> Nat .
  vars N M : Nat .
  eq plus(zero, M) = M .
  eq plus(succ(N), M) = succ(plus(N, M)) .
endm
```

Declarations and sentences are always terminated by periods, which should have white spaces before and after. Forgetting a terminal period or a white space before the period are two of the most common errors that Maude beginners make.

The signature of `PEANO-NAT` consists of one sort, `Nat`, and three operations, namely `zero`, `succ`, and `plus`. Sorts are declared with the keywords `sort` or `sorts`, and operations with `op` or `ops`.

The three operations have zero, one and two arguments, respectively, whose sorts are listed between the symbols `:` and `->`. Operations of zero arguments are also called *constants*, those of one argument are called *unary* and those of two *binary*. The result sort appears right after the symbol `->`.

We use `ops` when two or more operations of same arguments are declared together, to save space, and then we use white spaces to separate them:

```
  ops plus mult : Nat Nat -> Nat .
```

There are few special characters in Maude, and users are allowed to define almost any token or combination of tokens as operation names. If you use `op` in the above instead of `ops`, for example, then only one operation, called "`plus mult`", is declared.

The two equations in `PEANO-NAT` are properties, or constraints, that terms built with these operations must satisfy. Another way to look at equations is through the lenses of possible implementations of the specifications they define; in our case, any *correct* implementation of Peano natural numbers should satisfy the two equations. Equations are quantified universally with the variables they contain, and can be applied from *left-to-right* or from *right-to-left* in reasoning, which means that equational proofs may require exponential search, thus making them theoretically intractable. Maude provides limited support for equational reasoning.

`reduce`: Rewriting with Equations

When executing specifications, Maude regards all equations as *rewrite rules*, which means that they are applied only from left to right. Moreover, they are applied iteratively for as long as their left-hand-side terms match any subterm of the term to reduce. This way, any well-formed term can either be derived infinitely often, or be reduced to a *normal form*, which cannot be reduced anymore by applying equations as rewriting rules. Maude's command to reduce a term to its normal form using equations as rewrite rules is `reduce`, or simply `red`. Reduction will be made in the last defined module, which is `PEANO-NAT` in our case:

```
Maude> reduce plus(plus(succ(zero),succ(succ(zero))), succ(succ(succ(zero)))) .
rewrites: 6 in 0ms cpu (0ms real) (~ rewrites/second)
result Nat: succ(succ(succ(succ(succ(succ(zero))))))
```

Make sure commands are terminated with a period. Maude implements state of the art term rewriting algorithms, based on advanced indexing and pattern matching techniques. This way millions of rewrites per second can be performed, making `Maude` usable as a programming language in terms of performance.

Sometimes the results of reductions are repetitive and may be too large to read. To ameliorate this problem, Maude provides an operator attribute called `iter`, which allows to input and print repetitive terms more compactly. For example, if we replace the declaration of operation `succ` with

```
  op succ : Nat -> Nat  [iter] .
```

then Maude uses, e.g., `succ^3(zero)` as a shorthand for `succ(succ(succ(zero)))`. For example,

```
Maude> reduce plus(plus(succ(zero),succ^2(zero)), succ^3(zero)) .
result Nat: succ^6(zero)
```

Importation

Modules can be imported in several different ways. The difference between importation modes is subtle and semantical rather than operational, and it is not relevant in this book. Therefore, we only use the most general of them, `including`. For example, the following module extends `PEANO-NAT` with multiplication:

```
mod PEANO-NAT* is
  including PEANO-NAT .
  op mult : Nat Nat -> Nat .
  vars M N : Nat .
  eq mult(zero, M) = zero .
  eq mult(succ(N), M) = plus(mult(N, M), M) .
endm
```

It is safe to think of `including` as "copy and paste" the contents of the imported module into the importing module, with one exception: variable declarations are *not* imported, so they need to be redeclared.

We can now "execute programs" using features in both modules:

```
red mult(plus(succ(zero),succ(succ(zero))), succ(succ(succ(zero)))) .
```

The following is Maude's output:

```
rewrites: 18 in 0ms cpu (0ms real) (~ rewrites/second)
result Nat: succ^9(zero)
```

Even though this language is very simple and its syntax is ugly, it nevertheless shows a formal and executable definition of a language using equational logic and rewriting. Other languages or formal analyzers discussed in this book will be defined in a relatively similar manner, though, as expected, they will be more involved.

The Mixfix Notation and Parsing

The `plus` and `mult` operations defined above are meant to be written using the *prefix* notation in terms. Maude also supports the *mixfix* notation for operations (see Section 2.1.3), by allowing the user to write underscores in operation names as placeholders for their corresponding arguments.

```
op _+_ : Int Int -> Int .
op _! : Nat -> Nat .
op in_then_else_ : BoolExp Stmt Stmt -> Stmt .
op _?_:_ : BoolExp Exp Exp -> Exp .
```

Users can now also write terms taking advantage of the mixfix notation, for example `3 + 5`, in addition to the usual prefix notation, that is, `_+_(3,5)`.

Recall from Section 2.1.3 that, syntactically speaking, the mixfix notation has the same expressiveness as the context-free grammar notation. Therefore, the mixfix notation comes with the unavoidable *parsing* problem. For example, suppose that we replace the operations `plus` and `mult` in the modules above with their mixfix variants `_+_` and `_*_` (see also Exercise 29). Then the term `X + Y * Z`, with `X`, `Y`, `Z` arbitrary variables (or any terms) of sort `Nat`, admits two ambiguous parsings: `(X + Y) * Z` and `X + (Y * Z)`.

Maude provides a `parse` command, similar to `reduce` except that it only parses the given term:

```
Maude> parse X + Y .
Nat: X + Y
```

Maude generates a warning message whenever it detects more than one parsing of the given term:

```
Maude> parse X + Y * Z .
Warning: <standard input>, line 1: ambiguous term, two parses are:
X + (Y * Z)
-versus-
(X + Y) * Z
Arbitrarily taking the first as correct.
Nat: X + (Y * Z)
```

Similar warning messages are issued when ambiguous terms are detected in the specification (e.g., in equations). In general, we do not want to allow any parsing ambiguity in specifications or in terms to rewrite. One simple way to avoid ambiguities is to use parentheses to specify the desired grouping, for example:

```
Maude> parse X + (Y * Z) .
Nat: X + (Y * Z)
```

To reduce the number of parentheses, Maude allows us to assign precedences to mixfix operations declared in its modules, specifically as operator attributes in square brackets using the `prec` keyword. For example:

```
op _+_ : Nat Nat -> Nat [prec 33] .
op _*_ : Nat Nat -> Nat [prec 31] .
```

The lower the precedence the stronger the binding! As expected, now there is no parsing ambiguity anymore:

```
Maude> parse X + Y * Z .
Nat: X + Y * Z
```

To see how the term was parsed, set the "`print with parentheses`" flag on:

```
Maude> set print with parentheses on .
Maude> parse X + Y * Z .
Nat: (X + (Y * Z))
```

If displaying the parentheses is not sufficient, then disable the mixfix printing completely:

```
Maude> set print mixfix off .
Maude> parse X + Y * Z .
Nat: _+_(X, _*_(Y, Z))
```

Associativity, Commutativity and Identity Attributes

Some of the binary operations used in this book will be associative (A), commutative (C) or have an identity (I), or combinations of these. E.g., `_+_` is associative, commutative and has `0` as identity. All these can be added as attributes to operations when declared:

```
op _+_ : Int Int -> Int [assoc comm id: 0 prec 33] .
op _*_ : Int Int -> Int [assoc comm id: 1 prec 31] .
```

Note that each of the A, C, and I attributes are logically equivalent to appropriate equations, such as

```
eq A + (B + C) = (A + B) + C .
eq A + B = B + A .  ---> attention: rewriting does not terminate!
eq A + 0 = A .
```

When applied as rewrite rules, each of the three equations above have limitations. The associativity equation forces all the parentheses to be grouped to the left, which may prevent some other rules from applying. The commutativity equation may lead to non-termination when applied as a rewrite rule. The identity equation would only be able to simplify expressions, but not to add a `0` to an expression, which may be useful in some situations (we will see such situations shortly, in the context of lists). Maude's builtin support for ACI attributes addresses all the problems above. Additionally, the `assoc` attribute of a mixfix operation is also taken into account by Maude's parser, which hereby eliminates the need for some useless parentheses:

```
Maude> parse X + Y + Z .
Nat: X + Y + Z
```

An immediate consequence of the builtin support for the `comm` attribute, which allows rewriting with commutative operations to terminate, is that normal forms will be reported now *modulo commutativity*:

```
Maude> red X + Y + X .
rewrites: 0 in 0ms cpu (0ms real) (~ rewrites/second)
result Nat: X + X + Y
```

As seen, Maude picked to display some equivalent (modulo AC) of the original term (extracted from how the current implementation of Maude stores this term internally). There were 0 rewrites applied in the reduction above, because the internal rearrangements of terms according to the ACI attribute annotations do not count as rule applications.

Matching Modulo Associativity, Commutativity, and Identity

Here we discuss Maude's support for *ACI matching*, which is arguably one of the most distinguished and complex Maude features, and nevertheless the reason and the most important use of the ACI attributes.

We discuss ACI matching by means of a series of examples, starting with lists, which occur in many programming languages. The following module defines lists of integers with a membership operation `_in_`, based on AI (associative and identity) matching:

```
mod INT-LIST is including INT .
  sort IntList .
  subsort Int < IntList .
  op nil : -> IntList .
  op __ : IntList IntList -> IntList [assoc id: nil] .
  op _in_ : Int IntList -> Bool .
  var I : Int .  vars L L' : IntList .
  eq I in L I L' = true .
  eq I in L = false [owise] .
endm
```

We start by including the builtin INT module, which declares a sort Int and provides arbitrary large integers as constants of sort Int, together with the usual operations on these. The builtin module BOOL, which similarly declares a sort Bool and common Boolean operations on it, is automatically included in all modules, so it needs not be included explicitly. To see a an existing module, builtin or not, use the command

```
Maude> show module <NAME> .
```

For example, "show module INT ." will display the INT module. In the INT-LIST module above, note the subsort declaration "Int < IntList", which says that integers are also lists of integers. This, together with the constant nil and the concatenation operation __, can generate any finite list of integers:

```
Maude> parse 1 2 3 4 5 .
IntList: 1 2 3 4 5
Maude> red 1 nil 2 nil 3 nil 4 nil 5 6 7 nil .
rewrites: 0 in 0ms cpu (0ms real) (~ rewrites/second)
result IntList: 1 2 3 4 5 6 7
```

Note how the reduce command above eliminated all the unnecessary nil constants from the list, in zero rewrite steps, for the same reason as above: the internal rearrangements according to the ACI attributes do not count as rewrite steps.

The two equations defining the membership operation make use of AI matching. The first equation says that if we can match the integer I anywhere inside the list, then we are done. Since the list constructor __ was declared associative and with identity nil, Maude is mathematically allowed to bind the variables L and L' of sort IntList to any lists of integers, including the empty one. Maude indeed does this through its efficient AI matching algorithm. Equations with attribute owise are applied only when other equations fail to apply. Therefore, we defined the semantics of the membership operation only by means of AI matching, without having to implement any explicit traversal of the list. Here are some examples testing the semantics above:

```
Maude> red 3 in 2 3 4 .
result Bool: true
Maude> red 3 in 3 4 5 .
result Bool: true
Maude> red 3 in 1 2 4 .
result Bool: false
```

To define sets of integers (see, e.g., Exercise 30), besides likely renaming the sort IntList into IntSet, we would also need to declare the concatenation operation __ commutative; moreover, thanks to Maude's commutative matching, we can also replace the first equation by "eq I in I L = true ."

We next discuss a Maude definition of (partial finite-domain) maps (see Section 2.4.6 and Figure 2.7 for the mathematical definition). We assume that the Source and Target sorts are defined in separate modules SOURCE and TARGET, respectively; one may need to change these in concrete applications. The associativity, commutativity and identity equations in Figure 2.7 are replaced by corresponding Maude operational attributes. Note that the second equation defining the update operation takes advantage of Maude's owise attribute (explained above), so it departs from the more mathematical definition in Figure 2.7:

```
mod MAP is including SOURCE + TARGET .
  sort Map .
  op _|->_ : Source Target -> Map  [prec 0] .
  op empty : -> Map .
  op _,_ : Map Map -> Map  [assoc comm id: empty] .
  op _(_) : Map Source -> Target  [prec 0] .       --- lookup
  op _[_/_] : Map Target Source -> Map [prec 0] .  --- update
```

```
  var M : Map .  var A : Source .  var B B' : Target .
  eq (M, A |-> B)(A) = B .
  eq (M, A |-> B')[B / A ] = (M, A |-> B) .
  eq M[B / A] = (M, A |-> B) [owise] .
endm
```

If module SOURCE defines constants a, b, c, d, ..., of sort Source, and TARGET defines constants 1, 2, 3, 4, ..., of sort Target, then the following reduce commands work as shown:

```
Maude> red empty[1 / a][2 / b][3 / c] .
result Map: a |-> 1,b |-> 2,c |-> 3
Maude> red empty[1 / a][2 / b][3 / c][4 / a] .
result Map: a |-> 4,b |-> 2,c |-> 3
Maude> red empty[1 / a][2 / b][3 / c][4 / a](a) .
result Target: 4
Maude> red empty[1 / a][2 / b][3 / c][4 / a](d) .
result Target: (a |-> 4,b |-> 2,c |-> 3)(d)
```

Note that the last reduction above only updated the map, but it got stuck on the lookup of d. That is because we only have one equation defining lookup, which works only when the looked up element is in the domain of the map. Getting stuck terms as above may be acceptable in many applications, but, however, we sometimes want to report specific errors in such situations. Maude has several advanced foundational mechanisms to deal with errors, but they are non-trivial and we do not need them in this book. Instead, we can simply modify the MAP definition above to include a special undefined "value" and then explicitly use this value in equations where we mean that an error occurred:

```
op undefined : -> Map .
eq M(A) = undefined  [owise] .
```

Now the last reduction command above yields undefined. A particularly useful approach to deal with undefinedness in the context of programming language semantics, where a semantics builts upon several mathematical domains in which the syntax is interpreted, is to define the constant undefined to have a generic sort, say Domain, which is then subsorted to all sorts, including Source, Target, and Map. Then we can add the following to the module MAP to obtain partial finite-domain maps with support for undefinedness:

```
subsort Domain < Map .
eq M(A) = undefined  [owise] .
eq A |-> undefined = empty .
```

The last equation above is an optimization, allowing us to "garbage collect" the useless bindings in maps once they are explicitly "undefined" in certain elements. For example,

```
Maude> red empty[1 / a][2 / b][3 / c][4 / a][undefined / a] .
result Map: b |-> 2,c |-> 3
```

Pretty Printing

In the MAP example above, the bindings and the comma separating them may be hard to read when the maps are large. We may therefore want to pretty-print the reduced terms. Maude provides the format attribute for this purpose. For example, if we replace the operation declarations of _|->_ and _,_ with

```
op _|->_ : Source Target -> Map  [prec 0 format(d b o d)] .
op _,_ : Map Map -> Map  [assoc comm id: empty format(d sr! oss d)] .
```

then the former will always be displayed in color blue, while the second in bold red and preceded by one space and followed by two spaces. Each group of characters in the argument of `format` refers to a pointcut in the operation name; we have default pointcuts at the beginning and at the end of the operation name, as well pointcuts before and after any special token (underscore, comma, and the various kinds of parentheses and brackets). In each group of characters, `d` means "default" and is used alone to skip that pointcut and move to the next, `b` and `r` the colors blue and red, `o` means to revert to the original color and style, `s` means one space, and `!` means bold font. There are also indentation attributes, which we have not used here but we will use later in the book, such as: `+` and `-` to increment and decrement the global indent counter, respectively, `i` to print the number of spaces determined by indent counter, and `n` to print a newline.

Built-in Modules

Maude provides several builtin modules and has been designed in a way that existing modules can be easily changed and more modules can be added. It is likely that at this moment Maude's builtin modules are not identical to the homonymous ones when this book was written, and it also likely that new modules have been added since then. To avoid depending on particular versions of Maude, and also to avoid unfortunate naming conflicts with existing builtins which prevent us from naming programming language constructs as desired, in this book we actually define a custom version builtins, discussed in Section A.1. Nevertheless, some of Maude's current builtin modules make interesting use of ACI matching and are also present in our custom builtins, albeit using different names, so we briefly discuss them here.

As already discussed, the `INT` module provides a sort `Int` with arbitrarily large integers and common operations on them. All these are essentially hooked to C library functions through a special interface that Maude provides, which we do not discuss here but refer the interested reader to Section A.1 for details.

Similarly, there is a builtin module named `QID`, from "quoted identifiers", which provides a sort `Qid` together with arbitrary large quoted identifiers, as constants of sort `Qid`, such as the following: `'a`, `'b`, `` `abc123 ``, `'a-larger-identifier`, etc. These can be used as identifiers, e.g., as program variable names, in the programming languages that we define and execute using Maude.

Let us next discuss the module `BOOL`, which by default Maude includes in every other module (there are ways to disable the automatic inclusion, discussed in Section A.1). Besides the sort `Bool` with its two Boolean constants `true` and `false`, `BOOL` includes three important *polymorphic* operations and the usual Boolean operations. The polymorphic operations have the following syntax:

```
op if_then_else_fi : Bool Universal Universal -> Universal [...]
op _==_ : Universal Universal -> Bool [...]
op _=/=_ : Universal Universal -> Bool [...]
```

We excluded their specific attributes because they use advanced Maude features which are not necessary anywhere else in this book. Instead, we explain their behavior in words. The builtin sort `Universal` can be thought of as a generic sort, which can be instantiated with any concrete sort. Operation `if_then_else_fi` is strict in its first argument and lazy in its second and third arguments, that is, it only allows rewrites to take place in its first argument but not in the other two arguments. If the first argument rewrites to `true` then the conditional rewrites to its second argument, and if thefirst argument rewrites to `false` then the conditional rewrites to its third argument. We will discuss rewrite strategies in more depth later in this section. The other two operations correspond to operational equality and inequality of terms: the two terms are first rewritten to their normal forms and then those are compared modulo the existing operation ACI attributes. While operational equality implies logical equality, the other implication does not hold and tends to be a source of confusion, sometimes even among Maude experts: two terms t and t' may be provably equal using equational

reasoning, yet $t == t'$ may still rewrite to `false`. In this case, `false` only means that Maude was *not able* to show the two terms equal by rewriting them to their normal forms.

The definitions of the usual Boolean operations make interesting use of AC matching:

```
op _and_ : Bool Bool -> Bool [assoc comm prec 55] .
op _or_ : Bool Bool -> Bool [assoc comm prec 59] .
op _xor_ : Bool Bool -> Bool [assoc comm prec 57] .
op not_ : Bool -> Bool [prec 53] .
op _implies_ : Bool Bool -> Bool [prec 61 gather (e E)] .
vars A B C : Bool .
eq true and A = A .
eq false and A = false .
eq A and A = A .
eq false xor A = A .
eq A xor A = false .
eq A and (B xor C) = A and B xor A and C .
eq not A = A xor true .
eq A or B = A and B xor A xor B .
eq A implies B = not (A xor A and B) .
```

It can be shown that the equations above, when applied as rewrite rules, yield a decision procedure for propositional logic. Specifically, if we only consider `Bool` terms which are variables, any valid proposition rewrites to `true` and any unsatisfiable one to `false`; the remaining (satisfiable but invalid) propositions are rewritten to a canonical form consisting of an exclusive disjunction (`xor`) of conjunctions (`and`). We refer the interested reader to Section 2.3 for terminology and basic propositional logic results.

The attribute `gather(e E)` of `_implies_` tells the Maude parser that we want `_implies_` to be right associative, this way avoiding to add unnecessary parentheses in Boolean expressions. There is a similar attribute, `gather(E e)`, for left associativity. We do not discuss the gather attribute any further in this book.

Constructor versus Defined Operations

Recall the two equations of the module `PEANO-NAT`:

```
eq plus(zero, M) = M .
eq plus(succ(N), M) = succ(plus(N, M)) .
```

These equations constrain the three operations of the module, namely

```
op zero : -> Nat .
op succ : Nat -> Nat .
op plus : Nat Nat -> Nat .
```

But why did we write them in that particular style, which resembles a recursive definition of a *function* `plus` in terms of data-type *constructors* `zero` and `succ`? Intuitively, that is because we want `plus` to be completely *defined* in terms of `zero` and `succ`. Formally, this means that any term over the syntax `zero`, `succ`, and `plus` can be shown, using the given equations, equal to a term containing only `zero` and `succ` operations, that is, `zero` and `succ` alone are sufficient to build any Peano number.

While Maude at its core makes no distinction between operations meant to be data-type constructors and operations meant to be functions, it is still meaningful to distinguish the operations of a module into *constructor* and *defined* operations. Note, however, that it is the way we write the equations in the module, and only that, which makes the operations become constructors or defined. If we forget an equation dealing with a case (e.g., an intended constructor) for an operation intended to be defined, then that operation cannot

be always eliminated from terms, so technically speaking it is also a constructor. Unfortunately, there is no silver-bullet recipe on how to define "defined" operators, but essentially a good methodology is to define the operator's behavior on each intended constructor. That is what we did when we defined `plus` in `PEANO-NAT` and `mult` in `PEANO-NAT*`: we defined them on `zero` and on `succ`. In general, if `c1`, ..., `cn` are the intended constructors of a data-type, in order to define a new operation `d`, make sure that all equations of the form

```
eq d(c1(...)) = ...
...
eq d(cn(...)) = ...
```

are in the specification. If `d` has more arguments, then make sure that the above cases are listed for at least one of its arguments. This gives no guarantee (e.g., one can "define" `plus` as `plus(succ(M),N) = plus(succ(M),N)`), but it is a good enough principle to follow.

Let us demonstrate the above by defining several operations. Consider the following specification of lists:

```
mod INT-LIST is including INT .
  sort IntList .  subsort Int < IntList .
  op __ : Int IntList -> IntList [id: nil] .
  op nil : -> IntList .
endm
```

The two operations are meant to be constructors for lists, namely the empty list and adding an integer to the beginning of a list. Note, however, that the above contains *three* constructors for lists at first sight, because the subsorting of `Int` to `IntList` states that sole integers are also lists. Indeed, without the identity attribute of `__`, we would have to consider three cases when defining operations over lists. However, with the identity declaration Maude will internally identity integers `I` with lists `I nil`, so only two constructors are needed, corresponding to the two declared operations. Let us next define several important and useful operations on lists. Notice that the definition of each operator treats each of the two constructors separately.

The following defines the usual list length operator:

```
mod LENGTH is including INT-LIST .
  op length : IntList -> Nat .
  var I : Int .  var L : IntList .
  eq length(I L) = 1 + length(L) .
  eq length(nil) = 0 .
endm

red length(1 2 3 4 5) .   ***> should be 5
```

The following defines list membership, without speculating matching (in fact, this would not be possible anyway because concatenation is not defined associative as before):

```
mod IN is including INT-LIST .
  op _in_ : Int IntList -> Bool .
  vars I J : Int .  var L : IntList .
  eq I in J L = if I == J then true else I in L fi .
  eq I in nil = false .
endm

red 3 in 2 3 4 .   ***> should be true
red 3 in 3 4 5 .   ***> should be true
red 3 in 1 2 3 .   ***> should be true
red 3 in 1 2 4 .   ***> should be false
```

The next defines list append:

```
mod APPEND is including INT-LIST .
  op append : IntList IntList -> IntList .
  var I : Int .  vars L1 L2 : IntList .
  eq append(I L1, L2) = I append(L1, L2) .
  eq append(nil, L2) = L2 .
endm

red append(1 2 3 4, 5 6 7 8) .   ***> should be 1 2 3 4 5 6 7 8
```

Notice that append has two arguments and that we have picked the first one to define our cases on. One can still show that append is a defined operation, in that it can be eliminated by equational reasoning from any term of sort IntList. The following imports APPEND and defines an operation which reverses a list:

```
mod REV is including APPEND .
  op rev : IntList -> IntList .
  var I : Int .  var L : IntList .
  eq rev(I L) = append(rev(L), I) .
  eq rev(nil) = nil .
endm

red rev(1 2 3 4 5) .   ***> should be 5 4 3 2 1
```

The next module defines an operation, isort, which sorts a list of integers by insertion sort:

```
mod ISORT is including INT-LIST .
  op isort : IntList -> IntList .
  vars I J : Int .  var L : IntList .
  eq isort(I L) = insert(I, isort(L)) .
  eq isort(nil) = nil .
  op insert : Int IntList -> IntList .
  eq insert(I, J L) = if I > J then J insert(I,L) else I J L fi .
  eq insert(I, nil) = I .
endm

red isort(4 7 8 1 4 6 9 4 2 8 3 2 7 9) .   ***> should be 1 2 2 3 4 4 4 6 7 7 8 8 9 9
```

An auxiliary insert operation is also defined, which takes an integer and a sorted list and rewrites to a sorted list inserting the integer argument at its place in the list argument. Notice that this latter operation makes use of the builtn if_then_else_fi operation provided by the default BOOL module discussed above, as well as of the integer comparison operation ">" provided by the builtin module INT.

Let us now consider binary trees, where a tree is either empty or an integer with a left and a right subtree:

```
mod TREE is including INT .
  sort Tree .
  op ___ : Tree Int Tree -> Tree .
  op empty : -> Tree .
endm
```

We next define some operations on trees, following the tree structure given by the two constructors above.

The next operation mirrors a tree, i.e., it replaces left subtrees by the mirrored right siblings and vice-versa:

```
mod MIRROR is including TREE .
  op mirror : Tree -> Tree .
  vars L R : Tree .  var I : Int .
```

```
  eq mirror(L I R) = mirror(R) I mirror(L) .
  eq mirror(empty) = empty .
endm

red mirror((empty 3 (empty 1 empty)) 5 ((empty 6 empty) 2 empty)) .
***> should be (empty 2 (empty 6 empty)) 5 ((empty 1 empty) 3 empty)
```

Searching in binary trees can be defined as follows:

```
mod SEARCH is including TREE .
  op search : Int Tree -> Bool .
  vars I J : Int .  vars L R : Tree .
  eq search(I, L I R) = true .
 ceq search(I, L J R) = search(I, L) or search(I, R) if I =/= J .
  eq search(I, empty) = false .
endm

red search(6, (empty 3 (empty 1 empty)) 5 ((empty 6 empty) 2 empty)) .   ***> should be true
red search(7, (empty 3 (empty 1 empty)) 5 ((empty 6 empty) 2 empty)) .   ***> should be false
```

Note that we used a conditional equation above. Conditional equations are introduced with the keyword `ceq`, and their condition with the keyword `if`. There are several types of conditions in Maude, which we will discuss in the sequel, as needed. Here we used the simplest of them, namely a Bool term. To be faithful to rewriting logic (Section 2.5), we can regard a Boolean condition b as syntactic sugar for the equality b = `true`; in fact, Maude also allows us to write b = `true` instead of b. We can combine the first two equations above into an unconditional one, using an `if_then_else_fi` in its RHS (see Exercise 31).

We next define a module which imports both modules of trees and of lists on integers, and defines an operation which takes a tree and returns the list of all integers in that tree, in an infix traversal:

```
mod FLATTEN is
  including APPEND .
  including TREE .
  op flatten : Tree -> IntList .
  vars L R : Tree .  var I : Int .
  eq flatten(L I R) = append(flatten(L), I flatten(R)) .
  eq flatten(empty) = nil .
endm

red flatten((empty 3 (empty 1 empty)) 5 ((empty 6 empty) 2 empty)) .  ***> should be 3 1 5 6 2
```

Reduction Strategies

We sometimes want to inhibit the application of equations on some subterms, for executability reasons. For example, imagine a conditional expression construct `if_then_else_` in a pure language or calculus, i.e., one whose expression evaluation has no side-effects, say λ-calculus (this is discussed extensively in Section 4.5), whose semantics is governed by equations. While it is semantically safe to apply equations anywhere at any time, including inside the two branches of the conditional, for executability reasons we may prefer not to. Indeed, e.g., any reduction step applied in the negative branch would be a waste of computation if the condition turns out to evaluate to `true`. Worse, the wasteful reduction steps may lead to computational non-termination and resource exhaustion without giving the conditional a chance to evaluate its condition and then discard the non-terminating branch.

Because of reasons like above, many executable equational logic systems provide support for *reduction strategies*. In Maude, reduction strategies are given as operator `strat` attributes taking as argument sequences

of numbers. For example, the conditional operation would be assigned the strategy `strat(1 0)`, meaning that the first argument of the conditional is reduced to its normal form (the `1`), and *then* the conditional itself is allowed to be reduced (the `0`); since `2` and `3` are not mentioned, the second and the third argument of the conditional are never reduced while the conditional statement is still there. By default, operations of `n` arguments have strategy `strat(1 2...n 0)`, which yields a leftmost innermost reduction strategy.

Let us next discuss an interesting application of rewrite strategies in the context and lazy, infinite data-structures. One of the simplest such structure is the *stream*, that is, the infinite sequence. Specifically, we consider streams of integers, as defined in Figure 2.8. The key idea is to assign the stream construct `_:_`, which adds a given integer to the beginning of a given stream, the reduction strategy `strat(1 0)`. In other words, reduction is inhibited in the tail of streams built using the `_:_`. This allows us to have finite representations of infinite streams. To "observe" the actual elements of such stream structures, we define an operation `#` taking a natural number `N` and a stream `S`, and unrolling `S` until its first `N` elements become available. The module below defines this basic stream infrastructure, as well as some common streams and operations on them:

```
mod STREAM is including INT .
  sort Stream .
  op _:_ : Int Stream -> Stream [strat(1 0)] .
  var N : Int .  var S S' : Stream .
  op h : Stream -> Int .              eq h(N : S) = N .
  op t : Stream -> Stream .           eq t(N : S) = S .
  op # : Nat Stream -> Stream .   ***> #(N,S) displays the first N elements of stream S
  eq #(1, S) = h(S) : t(S) .
 ceq #(s(N), S) = h(S) : S' if S' := #(N, t(S)) .
  op zeros : -> Stream .              eq zeros = 0 : zeros .
  op ones : -> Stream .               eq ones = 1 : ones .
  op nat : Nat -> Stream .            eq nat(N) = N : nat(N + 1) .
  op zip : Stream Stream -> Stream .  eq zip(S, S') = h(S) : zip(S',t(S)) .
  op blink : -> Stream .              eq blink = zip(zeros,ones) .
  op add : Stream -> Stream .         eq add(S) = (h(S) + h(t(S))) : add(t(t(S))) .
  op fibonacci : -> Stream .          eq fibonacci = 0 : 1 : add(zip(fibonacci,t(fibonacci))) .
endm
```

The definition of the basic observers `h` (head) and `t` (tail) is self-explanatory. The definition of the general observer `#` involves a conditional equation, and that equation has a *matching* (`:=`) in its condition. In terms of equational logic, the matching symbol `:=` is nothing but the equality `=`; however, because of its special operational nature allowing us to introduce variables that do not appear in the LHS of the equation, Maude prefers to use a different symbol for it. The streams `zeros` and `ones`, of infinitely many 0 and 1 sequences, respectively, can be defined so compactly and then executed exactly because of the reduction strategy of `_:_`, which disallows the uncontrolled, non-terminating unrolling of these streams. However, we can observe as many elements of these streams as we wish, say 7, using controlled unrolling as follows:

```
red #(7, zeros) .                  ***> 0 : 0 : 0 : 0 : 0 : 0 : 0 : t(0 : zeros)
red #(7, ones) .                   ***> 1 : 1 : 1 : 1 : 1 : 1 : 1 : t(1 : ones)
```

Note how the reduction strategy enables the tails of the streams above, starting with their 8-th element, to stay unreduced, thus preventing non-termination.

The module above also defines the stream of natural numbers, an operation `zip` which interleaves two streams element by element, and operation `add` which generates a new stream by adding any two consecutive elements of a given stream, and finally two concrete streams defined using `zip`: one called `blink` which zips the streams `zeros` and `ones`, and one called `fibonacci` which contains the Fibonacci sequence (see also Exercise 25). Here are some sample reductions with these streams:

```
red #(7, nat(1)) .                  ***> 1 : 2 : 3 : 4 : 5 : 6 : 7 : t(7 : nat(1 + 7))
red #(7, blink) .                   ***> 0 : 1 : 0 : 1 : 0 : 1 : 0 : ...
red #(7, add(add(add(ones)))) .     ***> 8 : 8 : 8 : 8 : 8 : 8 : 8 : ...
red #(7, fibonacci) .               ***> 0 : 1 : 1 : 2 : 3 : 5 : 8 : ...
```

To save space, we did not show the remaining tail terms for the last three reductions.

Rewrite Rules

Until now we have only discussed one type of Maude sentences, its equations, and how to perform reductions with them regarded as rewrite rules, oriented from left to right. As seen in Section 2.5.1, rewrite logic has two types of sentences: equations and rewrite rules. Semantically, the rewrite rules establish transitions between equivalence classes of terms obtained using the equations. In other words, the distinction between equations and rewrite rules is that the former can be applied in any direction and do not count as computational steps in the transition systems associated to terms, while the latter can only be irreversibly applied from left to right and count as transitions. Because of the complexity explosion resulting from applying equations bidirectionally, Maude applies them in very restricted but efficient ways, which we have already discussed: equations like associativity, commutativity and identity are given by means of operator attributes and are incorporated within Maude's internal ACI rewrite algorithm, while other equations are only applied from left to right, same like the rewrite rules. While there is no visible distinction in Maude between equations and rewrite rules in terms of executability, several of Maude's tools, including the `search` command discussed below, treat them differently, so it is important to understand the difference between them.

The following module, inspired from the Maude manual, models a simple vending machine selling coffee for one dollar and tea for three quarters. The machine takes only one-dollar coins as input (`$`). For one dollar one can either buy one coffee, or one tea with a quarter rest. Assume that an external agent can silently and automatically change money as needed, that is, four quarters into a dollar (the viceversa is useless here); in other words, changing money does not count as a transaction in this simplistic vending machine model.

```
mod VENDING-MACHINE is
    sorts Coin Item State .
    subsorts Coin Item < State .
    op __ : State State -> State [assoc comm id: null] .
    op null : -> State .
    op $ : -> Coin      [format (r! o)] .
    op q : -> Coin      [format (r! o)] .
    op tea : -> Item    [format (b! o)] .
    op coffee : -> Item [format (b! o)] .

    rl $ => coffee .
    rl $ => tea q .
    eq q q q q = $ .
endm
```

All the coins and all the items that one has at any given moment forms a `State`. The two rules above model the two actions that the machine can perform, and the equation models the money changing agent.

A first distinction between equations and rules can be seen when using the `reduce` command:

```
Maude> red $ q q q .
result State: $ q q q
```

Note that the equation cannot reduce this state any further when applied from left to right, and that no rewrite rules were applied. Indeed, the command `reduce` only applies equations, and only from left-to-right. To apply both equations *and* rewrite rules, we have to use the command `rewrite` or its shortcut `rew`:

```
Maude> rew $ q q q .
result State: q q q coffee
```

Maude chose to apply the first rule and the resulting state cannot be rewritten or reduced anymore, so we are stuck with three useless quarters. If Maude had chosen the second rule, then we could have bought both a coffee and a tea with our money. To see all possible ways to rewrite a given term, we should use the `search` command instead of `rewrite`:

```
Maude> search $ q q q =>! S:State .
Solution 1 (state 1)
S:State --> q q q coffee
Solution 2 (state 3)
S:State --> tea coffee
Solution 3 (state 4)
S:State --> q tea tea
```

The `search` command takes a term and a pattern and attempts to systematically (in breadth-first order) apply the rewrite rules on the original term in order to match the pattern. In our case, the pattern is a state variable `S`, so all states match it. The term and the pattern are separated by a decorated arrow. Different decorations mean different things. The `!` above means that we are interested only in normal forms. Indeed, the above search command has precisely three solutions, as reported by Maude. Another useful decoration is `*`, which shows all intermediate states, not only the normal forms:

```
Maude> search $ q q q =>* S:State .
Solution 1 (state 0)
S:State --> $ q q q
Solution 2 (state 1)
S:State --> q q q coffee
Solution 3 (state 2)
S:State --> $ tea
Solution 4 (state 3)
S:State --> tea coffee
Solution 5 (state 4)
S:State --> q tea tea
```

Notice that we never see four quarters in a state in the solutions above, in spite of the `*` decoration; the equation automatically changed them into a dollar. Remember: *equations do not count as transitions*, so their application is not visible in the transition system explored by `search`. Another way to think of it is *rewrite rules apply modulo equations*; that is, equations structurally rearrange the term so that rules match and apply. Yet another way to think about it is that equations take time zero to apply, i.e., they apply instantaneously no matter how many they are, while rules take time one. In terms of performance, rules are slightly slower in Maude because they require more infrastructure to be maintained in terms, but that should not be the deciding factor when choosing whether a sentence should be an equation or a rule. One typical criterion for deciding what is an equation and what is a rule is that computations performed with the former are meant to be deterministic, while rewrite rules can lead to non-deterministic behaviors (like our two rules above).

One of Maude's major strengths, in addition to its efficient support for rewriting modulo ACI, is its capability to perform reachability analysis in conditions of rules. Consider, for example, the following extension of our vending machine model (`S` and `S'` are variables of sort `State`):

```
 op both? : State -> Bool .
crl both?(S) => true if S => coffee tea S' .
```

The conditional rule rewrites `both?(S)` to `true` when `S` rewrites to at least one coffee and one tea. However, to check the condition, exhaustive search may be needed. Otherwise, wrong normal forms may be reported.

```
Maude> rew both?(q q q q q q q q) .
result Bool: true
```

Maude correctly reported `true` above; however, without search in the condition one could have wrongly reported the term stuck, for example if one would have bought two coffees from the two dollars in the condition instead of one coffee and one tea.

As expected, the `search` command also works with conditional rules:

```
Maude> search both?(q q q q q q q q) =>* B:Bool .
Solution 1 (state 0)
B:Bool --> both?($ $)
Solution 2 (state 1)
B:Bool --> true
Solution 3 (state 2)
B:Bool --> both?($ coffee)
Solution 4 (state 3)
B:Bool --> both?($ q tea)
Solution 5 (state 4)
B:Bool --> both?(coffee coffee)
Solution 6 (state 5)
B:Bool --> both?(q tea coffee)
Solution 7 (state 6)
B:Bool --> both?(q q tea tea)
```

Interestingly, note that many other solutions besides `true` have been reported above, some of them in normal form (use `!` instead of `*` in the search command to see only those in normal form). That is because the rules of the original vending machine applied inside the argument of `both?`, which is something that we did not intend to happen when we wrote the conditional rule above. To prohibit rewrite rules from applying in some arguments of an operation, we have to use the `frozen` operator attribute with corresponding arguments:

```
op both? : State -> Bool [frozen(1)] .
```

Unlike the `strat` attribute which gives permission to reductions inside operator arguments, the `frozen` attribute takes permission to rewrites inside operator arguments. The `strat` attribute only works with equations and is ignored by rewrite rules, while the `frozen` attribute only works with rewrite rules and is ignored by equations. For example, the above search command still reports two solutions:

```
Maude> search both?(q q q q q q q q) =>* B:Bool .
Solution 1 (state 0)
B:Bool --> both?($ $)
Solution 2 (state 1)
B:Bool --> true
```

The first solution is an artifact of `frozen` allowing the equation to apply within the argument of `both?`. If one wants to prohibit both rules and equations, then one should use both `frozen` and `strat` attributes:

```
op both? : State -> Bool [frozen(1) strat(0)] .
```

Rules can have multiple conditions and the conditions can share variables, e.g., (`I`,`I'` of sort `Item`)

```
 op solve : State -> State [frozen(1) strat(0)] .
crl solve(S) => S' if S => I I I S' /\ S => I' I' I' I' S' .
```

The rule above says that `solve(S)` rewrites to `S'` when `S'` is a rest that can be obtained from `S` both after buying three identical items and after buying four identical items. Common sense tells us that the three identical items must be coffee and the four identical items must be tea, and that `S'` is `S` minus the three dollars spent to buy any of these groups of identical items. But Maude does not have this common sense, it needs to search. Specifically, it will do search in the first condition until three identical items are reached, then it does search in the second condition until four identical items are found with the same rest `S'` as in the first search; if this is not possible, then it backtracks and searches for another match of three identical items in the first rule, and so on and so forth until the entire state-space is exhausted (if finite, otherwise possibly forever).

Interestingly, although either three coffees or four teas cost three dollars, we cannot buy each of these with three dollars:

```
Maude> search solve($ $ $) =>! S .
Solution 1 (state 0)
S --> solve($ $ $)
```

The term `solve($ $ $)` is in normal form because there is no way to satisfy the second condition of the conditional rule above. However, if we have one quarter in addition to the three dollars, then we can satisfy the second condition of the rule, too, because we can first buy three teas getting three quarters back, which together with the additional quarter can be changed into one dollar, which gives us one more tea and a quarter back. So from three dollars and a quarter we can buy either three coffees or four teas, with a quarter rest:

```
Maude> rew solve($ $ $ q) .
result Coin: q
Maude> search solve($ $ $ q) =>! S .
Solution 1 (state 1)
S --> q
```

Therefore, both the rewrite and the search commands above have solved the double condition of the conditional rule. Moreover, the search command tells us that there is only one solution to the conditional rule's constraints.

When giving semantics to programming languages using rewrite logic, conditional rules will be used to reduce the semantic task associated to a language construct to similar semantic tasks associated to its arguments. Since some language constructs have a non-deterministic behavior, the search capability of Maude has a crucial role. In spite of the strength and elegance of Maude's conditional rules, note, however, that they are quite expensive to execute. Indeed, due to the combined variable constraints resulting from the various conditions, in the worst case there is no way to avoid an exhaustive search of the entire state-space in rules' conditions. Additionally, each rule used in the search-space of a condition can itself be conditional, which may itself require an exhaustive search to solve its condition, and so on and so forth. All this nested search process is clearly expensive. It is therefore highly recommended to avoid conditional rules whenever possible. As seen in Chapter 3, some semantic styles cannot avoid the use of conditional rules, but others can.

Turing Machines in Maude

Section 2.5.3 showed how to faithfully represent Turing machines in rewrite logic, in a way that any computational step in the original Turing machine corresponds to precisely one rewrite step in its (rewrite theory) representation and viceversa. Here we show how such rewrite theories can be defined and executed in Maude. Since the Turing machine transitions are represented as rewrite rules and not as equations, non-deterministic Turing machines can be represented as well following the same approach and without any additional complexity. Following the model in Section 2.5.3, we first discuss the representation based on lazy reduction strategies and then the unrestricted representation. For the lazy one, the idea is to define the infinite tape of zeros lazily, as we did when we defined the various kinds of streams above:

```
mod TAPE is
  sorts Cell Tape .
  ops 0 1 : -> Cell .
  op _:_ : Cell Tape -> Tape [strat(1 0)] .
  op zeros : -> Tape .
  eq zeros = 0 : zeros .
endm
```

Thanks to the `strat(1 0)` reduction strategy of the `_:_` construct above, the expanding equation of `zeros` only applies when `zeros` is on a position different from the tail of a stream/tape; in particular, it cannot be applied to further expand the `zeros` in its RHS.

Now we can define any Turing machine $\mathcal{M}$ in Maude using the approach in Figure 2.10 by importing `TAPE`, defining operations "`q : Tape Tape -> Tape`" for all $q \in Q$, and adding all the rules corresponding to the Turing machine's transition function (as explained in Figure 2.10). For example, the Maude representation of the Turing machine in Figure 2.1 that computes the successor function is:

```
mod TURING-MACHINE-SUCC is including TAPE .
  sort Configuration .
  ops qs qh q1 q2 : Tape Tape -> Configuration .
  var L R : Tape .  var B : Cell .
  rl qs(L, 0 : R) => q1(0 : L, R) .
  rl q1(L, 0 : R) => q2(L, 1 : R) .
  rl q1(L, 1 : R) => q1(1 : L, R) .
  rl q2(L, 0 : R) => qh(L, 0 : R) .
  rl q2(B : L, 1 : R) => q2(L, B : 1 : R) .
endm
```

Now we can "execute" the Turing machine using Maude's rewriting:

```
Maude> rew qs(zeros, 0 : 1 : zeros) .
result Configuration: qh(0 : zeros, 0 : 1 : 1 : zeros)
Maude> rew qs(zeros, 0 : 1 : 1 : 1 : 1 : zeros) .
result Configuration: qh(0 : zeros, 0 : 1 : 1 : 1 : 1 : 1 : zeros)
```

Recall from Section 2.2.1 that a natural number input n is encoded as $n + 1$ bits of 1 following the start cell where the head of the machine is initially, which holds the bit 0. So the first rewrite command above rewrites the natural number 0 to its successor 1, while the second rewrites 3 to 4. Note that the self-expanding equation of `zeros` is not applied backwards, so resulting streams/tapes of the form `0 : zeros` are not compacted back into `zeros`. One needs specific equations to do so, which we do not show here (see Exercise 34).

Non-deterministic Turing machines can be defined equally easily. For example, the machine non-deterministically choosing to yield the successor or not when it reaches the end of the input, also discussed in Section 2.5.3, can be obtained by adding the rule

```
  rl q1(B : L, 0 : R) => q2(L, B : 0 : R) .
```

Of course, `search` is needed now in order to explore all the non-deterministic behaviors:

```
Maude> search qs(zeros, 0 : 1 : 1 : 1 : 1 : zeros) =>! C:Configuration .
Solution 1 (state 16)
C:Configuration --> qh(0 : zeros, 0 : 1 : 1 : 1 : 1 : 0 : zeros)
Solution 2 (state 18)
C:Configuration --> qh(0 : zeros, 0 : 1 : 1 : 1 : 1 : 1 : zeros)
```

The unrestricted representation of Turing machines in rewrite logic discussed in Section 2.5.3 (Figure 2.11) can also be easily defined and then executed in Maude. In fact, this unrestricted representation has the advantage that it requires no special support for reduction strategies from the underlying rewrite system, so it should be easily adaptable to other rewrite systems than Maude. Since the self-expanding equation of *zeros* is not needed anymore, we can now define the tape as a plain algebraic signature:

```
mod TAPE is
  sorts Cell Tape .
  ops 0 1 : -> Cell .
  op _:_ : Cell Tape -> Tape .
  op zeros : -> Tape .
endm
```

The two new equations in Figure 2.11 can be defined generically as follows, for any Turing machine state:

```
mod TURING-MACHINE is including TAPE .
  sorts State Configuration .
  op _(_,_) : State Tape Tape -> Configuration .
  var S : State .  var L R : Tape .
  eq S(zeros,R) = S(0 : zeros, R) .
  eq S(L,zeros) = S(L, 0 : zeros) .
endm
```

Particular Turing machines can now be defined by including the module TURING-MACHINE above and adding specific states and rules. For example, here is the one calculating the successor already discussed above:

```
mod TURING-MACHINE-SUCC is including TURING-MACHINE .
  ops qs qh q1 q2 : -> State .
  var L R : Tape .  var B : Cell .
  rl qs(L, 0 : R) => q1(0 : L, R) .
  rl q1(L, 0 : R) => q2(L, 1 : R) .
  rl q1(L, 1 : R) => q1(1 : L, R) .
  rl q2(L, 0 : R) => qh(L, 0 : R) .
  rl q2(B : L, 1 : R) => q2(L, B : 1 : R) .
endm
```

The Post Correspondence Problem in Maude

We here show how to define in Maude the rewrite theory in Section 2.5.3 which allows to reduce the Post correspondence problem to rewrite logic reachability. We define strings as AI sequences of symbols, but for output reasons we prefer tiles to be triples instead of just pairs of strings, where the additional string acts as the label of the tile. To easily distinguish labels from each other, we prefer to technically work with strings of natural numbers instead of bits, although we will only use the numbers 0 and 1 in strings not meant to serve as labels. The module below is self-explanatory:

```
mod PCP is including NAT .
  sorts Symbol String .  subsort Nat < Symbol < String .
  sorts Tile Tiles .     subsort Tile < Tiles .
  op . : -> String .                                          --- empty string
  op __ : String String -> String [assoc id: .] .             --- string concatenation
  op _[_,_] : String String String -> Tile [prec 3] .         --- first string is the label
  op __ : Tiles Tiles -> Tiles [assoc comm] .                 --- concatenation of tiles
  var L L' R R' S S' : String .
  rl L[R,S] L'[R',S'] => L[R,S] L'[R',S'] (L L')[R R', S S'] .  --- the only rule
endm
```

So a tile has the form *label*$[\alpha,\beta]$, where *label* is the label of the tile and α and β are the tile's two strings. The unique rule matches any two tiles and adds their concatenation to the pool, without removing them. This way, we can start with any set of tiles and eventually reach any combination of them. Obviously this rewrite theory does not terminate. We should only use the `search` command here. Specifically, we should search for patterns containing a tile of the form *label*$[\gamma,\gamma]$. Any term of sort `Tiles` matching such a pattern indicates that a successful combination of the original tiles has been found; moreover, *label* will be bound to the desired sequence of labels of the combined tiles and γ to the combined string. For example, the following shows that the specific PCP problem mentioned in Section 2.2.2 is solvable, as well as a solution:

```
Maude> search[1] 1[0, 1 0 0] 2[0 1, 0 0] 3[1 1 0, 1 1] =>*  L[S,S] Ts:Tiles .
L --> 3 2 3 1
S --> 1 1 0 0 1 1 1 0 0
Ts:Tiles --> 1[0,1 0 0] 2[0 1,0 0] 3[1 1 0,1 1] (2 3)[0 1 1 1 0,0 0 1 1] ...
```

It is important to use `search[1]` above, because otherwise Maude will continue to search for all solutions and thus will never terminate. The option `[1]` tells Maude to stop searching after finding one solution.

Since the PCP problem is undecidable, we can conclude that Maude's search is undecidable in general (which is not a surprise, but we now have a formal proof).

2.5.7 Exercises

Exercise 27. *Eliminate the two generic equations in Figure 2.11 as discussed below Theorem 11 and prove that the resulting rewrite theory is confluent and captures the computation in $\mathcal{M}$ faithfully, step-for-step.*

Exercise 28. *Prove that any correct implementation of* `PEANO-NAT` *(Section 2.5.6) should satisfy the property*

```
plus(succ(succ(zero)), succ(succ(succ(zero)))) =
plus(succ(succ(succ(succ(zero)))), succ(zero)).
```

Exercise 29. *Rewrite* `PEANO-NAT` *and* `PEANO-NAT*` *(Section 2.5.6) using Maude's mixfix notation for operations. What happens if we try to reduce an expression containing both* `_+_` *and* `_*_` *without parentheses?*

Exercise 30. *Define a Maude module called* `INT-SET` *specifying sets of integers with membership, union, intersection, difference (elements in the first set but not in the second), and symmetric difference (elements in any of the two sets but not in the other).*

Exercise 31. *Define the* `search` *operation in module* `SEARCH` *(Section 2.5.6) with only two unconditional equations, using the built-in* `if_then_else_fi`.

Exercise 32. *Recall module* `FLATTEN` *(Section 2.5.6) which defines and infix traversal operation on binary trees. Do the same for prefix and for postfix traversals.*

Exercise 33. *Write a Maude module that uses binary trees as defined in module* `TREE` *(Section 2.5.6) to sort lists of integers. You should define an operation* `btsort : IntList -> IntList`*, which sorts the argument list of integers (like the* `isort` *operation in module* `ISORT` *in Section 2.5.6). In order to define* `btsort`*, define another operation,* `bt-insert : IntList Tree -> Tree`*, which inserts each integer in the list at its place in the tree, and also use the* `flatten` *operation already defined in module* `FLATTEN`.

Exercise 34. *When executing Turing machines in Maude as shown in Section 2.5.6, we obtain final configurations which contain (sub)streams/tapes of the form* `0 : zeros`*. While these are semantically equal to* `zeros`*, they decrease the readability of the final Turing machine configurations. Add generic equations to*

canonicalize the final configurations by iteratively replacing each sub-term `0 : zeros` *with* `zeros`*.*
Hint: A special reduction strategy may be needed for the operation `qh` *(or the configuration construct, respectively), to inhibit the application of the self-expanding equation of* `zeros`*.*

Exercise 35. *Define in Maude the Turing machines corresponding to the addition, the multiplication, and the power operations on natural numbers in Exercise 12. Do it using three different approaches: (1) using the infinite stream* `zeros`*, following the representation in Figure 2.10; (2) without using infinite streams but mimicking them with the by-need expansion using the two equations in Figure 2.11; (3) without any equations, following the style suggested right after Theorem 11, at the expense of adding more rules.*

Exercise 36. *Use the Maude definition of the Post correspondence problem in Section 2.5.6 to calculate the least common multiplier of two natural numbers. The idea here is to create an appropriate set of tiles from the two numbers so that the solution to the search command contains the desired least common multiplier.*

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