A Semantic Approach to Interpolation

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Craig interpolation and Computer Science

- Interpolation [Craig, 1957] is a landmark result in FOL:
  If $\Gamma_1 \vdash \Gamma_2$, then there is an *interpolant* Γ such that:
  - $\Gamma_1 \vdash \Gamma$, $\Gamma \vdash \Gamma_2$
  - All symbols of Γ occur in *both* Γ_1 and Γ_2
- Computer Science applications:
  - SAT-based model checking [McMillan, CAV'03], theorem proving
    If $\Gamma_1 \wedge \Gamma_2$ unsat then there exists Γ on their common symbols such that $\Gamma_1 \implies \Gamma$ and $\Gamma \wedge \Gamma_2$ unsat
  - Theorem proving
    Split proof in two by finding an interpolant
  - Specification theory
    Existence of interpolants ensures soundness of structured module flattening

Approaches to proving interpolation

- Syntactical
  - Always constructs an interpolant in FOL
  - Does not usually give specialized answers for sublogics
- Our (semantical) approach:
  - General technique for proving interpolation, via Birkhoff-like axiomatizability results
  - Captures the type of the interpolant more precisely
  - Model theoretical (non-constructive) approach
  - Applicable to amalgamations of FOL sub-logics
- Previous work using semantic techniques

  [Rodenburg, 1991], [Roşu&Goguen, 2000], [Diaconescu, 2004]

Craig interpolation - diagrammatically

If $\Gamma_1 \vdash \Gamma_2$, then there is an *interpolant* Γ such that:

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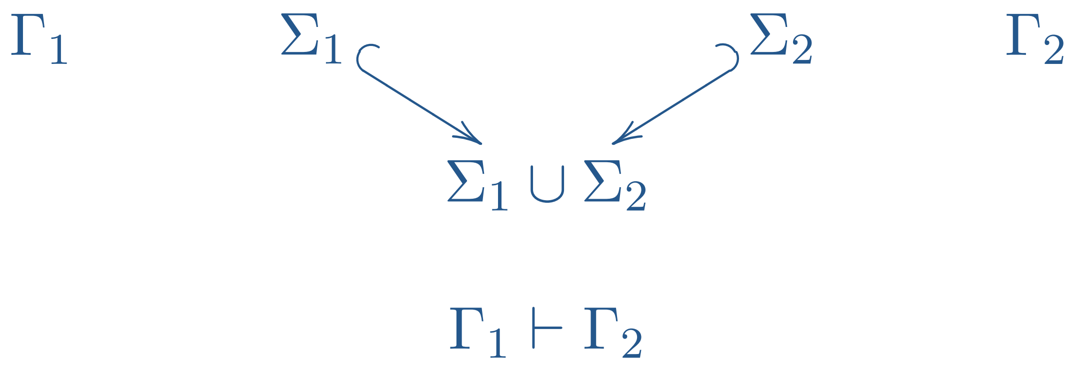

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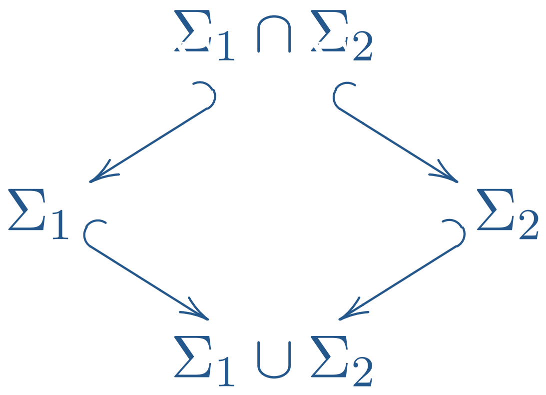
Γ

$\Gamma_1 \vdash \Gamma$  $\Gamma \vdash \Gamma_2$

$\Gamma_1 \vdash \Gamma_2$

Craig interpolation - a semantic point of view

$\Gamma_1 \vdash \Gamma_2$ iff each model of Γ_1 is also a model for Γ_2

Syntactically

$$\begin{array}{cccc} \Gamma_1 & \Sigma_1 & \Sigma_2 & \Gamma_2 \\ & \searrow & \swarrow & \\ & \multicolumn{2}{c}{\Sigma_1 \cup \Sigma_2} & \\ & \multicolumn{2}{c}{\Gamma_1 \vdash \Gamma_2} & \end{array}$$

Semantically

$$\begin{array}{cccc} {\Gamma_1}^* & Mod(\Sigma_1) & Mod(\Sigma_2) & {\Gamma_2}^* \\ & \nwarrow {\restriction_{\Sigma_1}} & {\restriction_{\Sigma_2}} \nearrow & \\ & \multicolumn{2}{c}{Mod(\Sigma_1 \cup \Sigma_2)} & \\ & \multicolumn{2}{c}{{\restriction_{\Sigma_1}}^{-1}({\Gamma_1}^*) \subseteq {\restriction_{\Sigma_2}}^{-1}({\Gamma_2}^*)} & \end{array}$$

Craig interpolation - a semantic point of view

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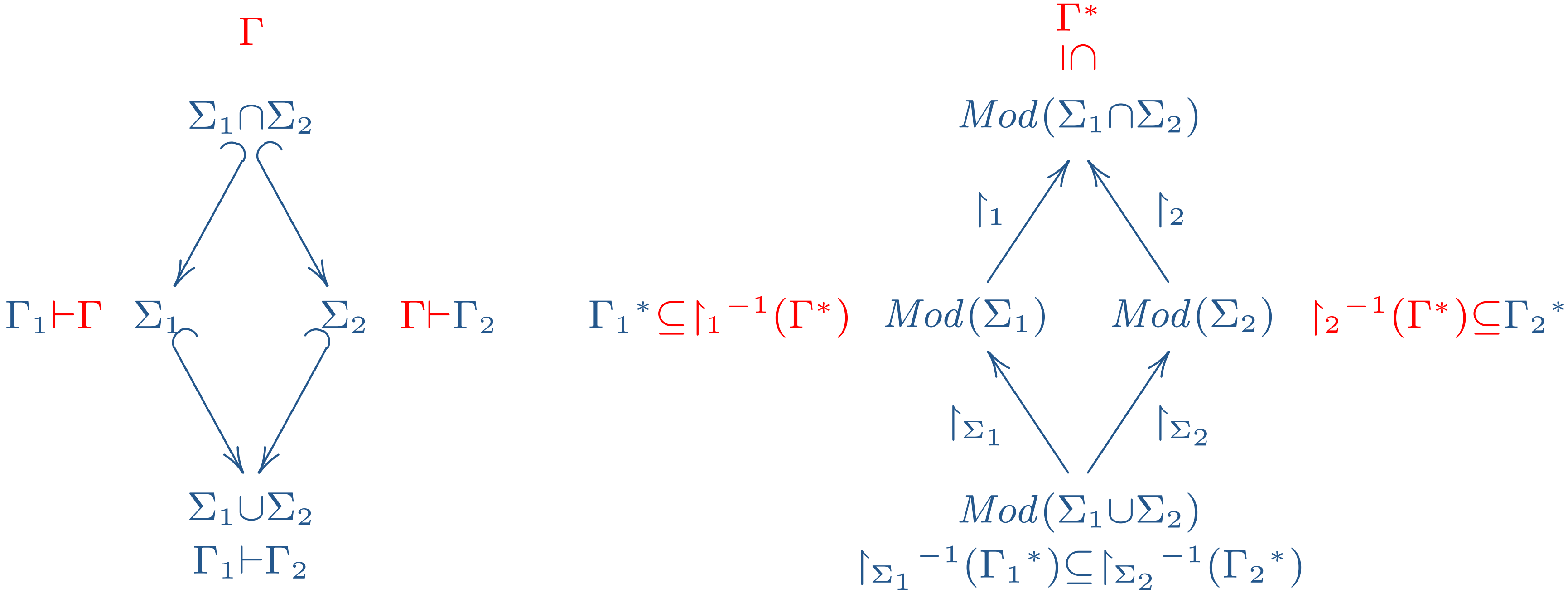
Syntactically Semantically



Concept stated in terms of inclusions between (axiomatizable) classes of models

Abstracting away the logic

Birkhoff like axiomatizability results characterize axiomatizability by means of closure properties

- *Birkhoff:* A class is equational iff it is closed under subobjects, homomorphic images, and products
- *Keisler-Shelah:* A class is elementary iff it is closed under ultraproducts and ultraradicals

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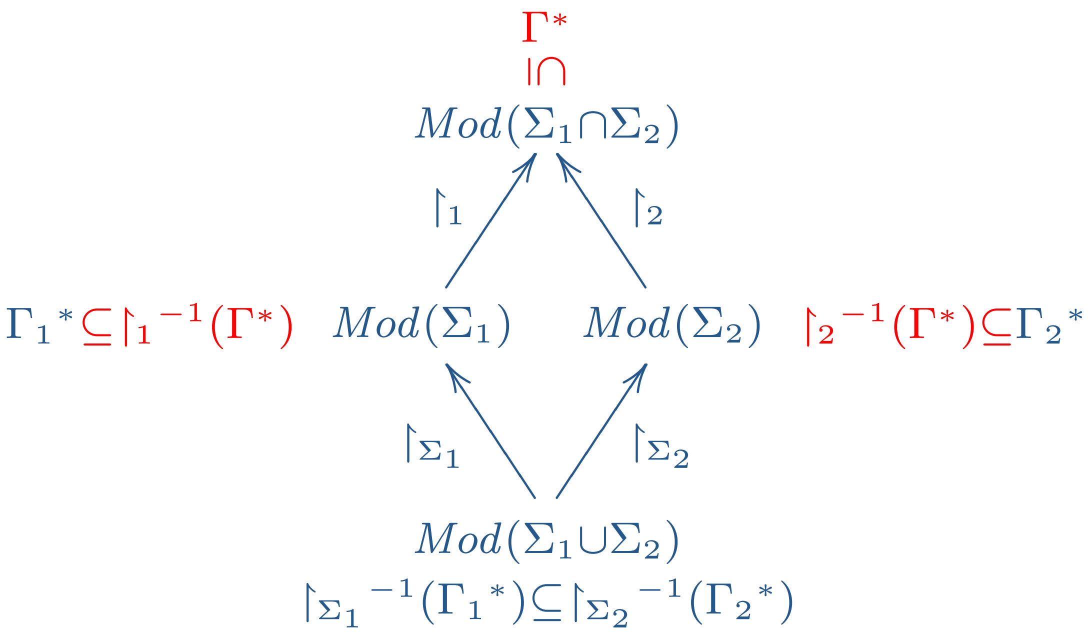
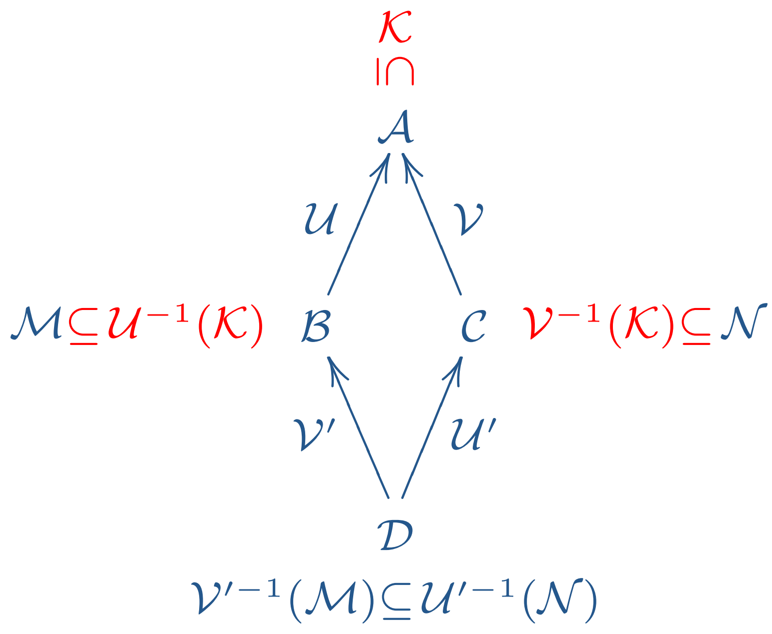


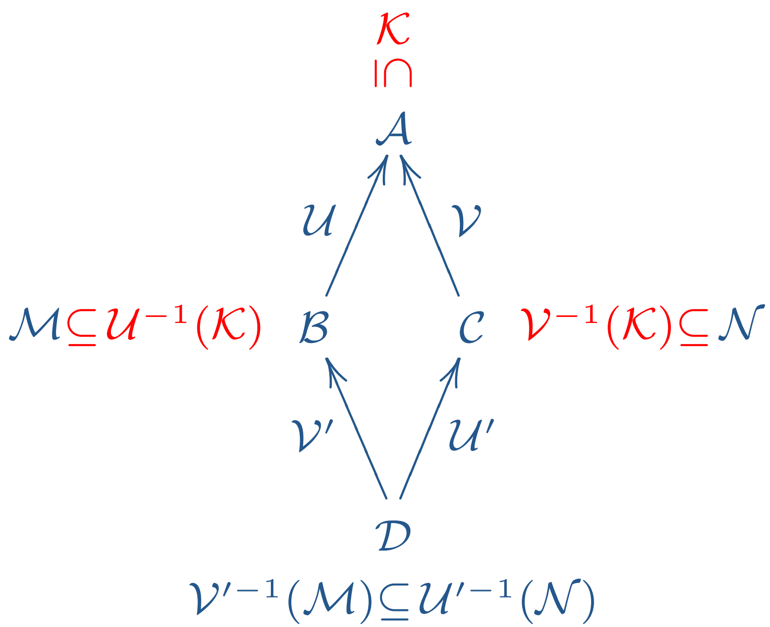


Our approach: Divide&Conquer

1. Identify the candidates for interpolants by solving the abstract problem

$$\mathcal{I}(\mathcal{M},\mathcal{N}) = [\mathcal{U}(\mathcal{M}), \overline{\mathcal{V}(\overline{\mathcal{N}})}]$$

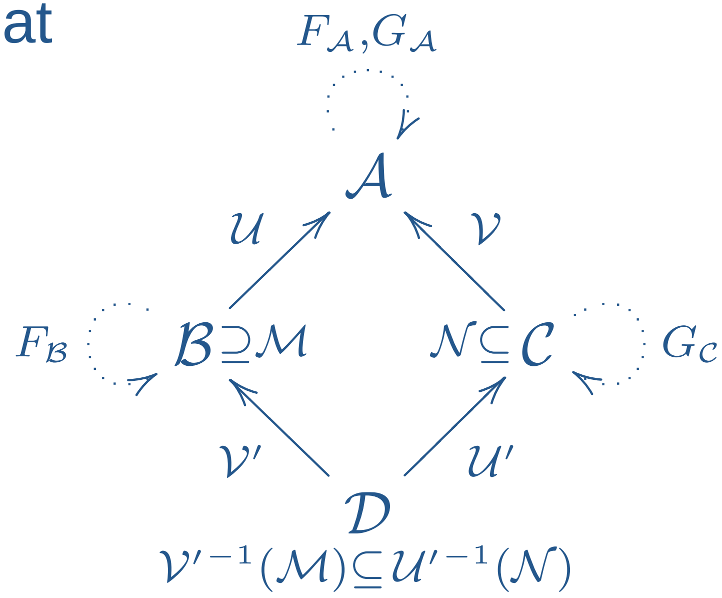
2. Give general conditions under which closure properties of the original classes are preserved by $\mathcal{U}$ and $\mathcal{V}$, thus validating the candidate



Main theorem

In any weak amalgamation square such that

- $F_{\mathcal{A}}$, $F_{\mathcal{B}}$ are arbitrary operators
- $G_{\mathcal{A}}$, $G_{\mathcal{B}}$ are closure operators
- $F_{\mathcal{A}}; G_{\mathcal{A}}; F_{\mathcal{A}} = F_{\mathcal{A}}; G_{\mathcal{A}}$
- $\mathcal{U}$ preserves fixed points of F
- $\mathcal{V}$ lifts G, that is, for all $\mathcal{X}$,
  $\mathcal{V}^{-1}(G_{\mathcal{A}}(\mathcal{X})) \subseteq G_{\mathcal{C}}(\mathcal{V}^{-1}(\mathcal{X}))$



if $\mathcal{M}$ is a fixed point of $F_{\mathcal{B}}$ and $\mathcal{N}$ one of $G_{\mathcal{C}}$,
then K is a fixed point of both $F_{\mathcal{A}}$ and $G_{\mathcal{A}}$.

Important: Since closure properties are lifted from both sizes, sometimes the interpolant is found in the intersection logic.

E.g.: Universal Horn clauses and positive sentences yield universal quantified atoms

Known results as instances of the theorem (I)

For $\Gamma_1 \vdash \Gamma_2$, find Γ such that $\Gamma_1 \vdash \Gamma$ and $\Gamma \vdash \Gamma_2$.
Γ should only have symbols common to Γ_1 and Γ_2.

[Craig 1957]

- Γ_1 - First order sentence
- Γ_2 - First order sentence
- Γ - First order sentence

Known results as instances of the theorem (II)

For $\Gamma_1 \vdash \Gamma_2$, find Γ such that $\Gamma_1 \vdash \Gamma$ and $\Gamma \vdash \Gamma_2$.
Γ should only have symbols common to Γ_1 and Γ_2.

[Rodenburg 1991]

- Γ_1 - Equations (conditional or not)
- Γ_2 - Equations (conditional or not)
- Γ - Equations (conditional or not)

New results as instances of the theorem (I)

For $\Gamma_1 \vdash \Gamma_2$, find Γ such that $\Gamma_1 \vdash \Gamma$ and $\Gamma \vdash \Gamma_2$.
Γ should only have symbols common to Γ_1 and Γ_2.

Non-trivial generalization of [Rodenburg 1991]:

- Γ_1 - First order sentences
- Γ_2 - Equations (with or without conditions)
- Γ - Equations (with or without conditions)

New results as instances of the theorem (II)

For $\Gamma_1 \vdash \Gamma_2$, find Γ such that $\Gamma_1 \vdash \Gamma$ and $\Gamma \vdash \Gamma_2$.
Γ should only have symbols common to Γ_1 and Γ_2.

In relational first order logic:

- Γ_1 - Universal Horn clauses
- Γ_2 - Positive $(\wedge, \vee, \exists, \forall)$ sentences
- Γ - Universal atoms

The interpolant's type refines both the original sentences' types.

New results as instances of the theorem (II)

For $\Gamma_1 \vdash \Gamma_2$, find Γ such that $\Gamma_1 \vdash \Gamma$ and $\Gamma \vdash \Gamma_2$.
Γ should only have symbols common to Γ_1 and Γ_2.

In relational first order logic:

- Γ_1 - Universal Horn clauses — conditional equations
- Γ_2 - Positive $(\wedge, \vee, \exists, \forall)$ sentences
- Γ - Universal atoms — equations

Open question: What if we allow functional symbols?

New results as instances of the theorem (III)

For $\Gamma_1 \vdash \Gamma_2$, find Γ such that $\Gamma_1 \vdash \Gamma$ and $\Gamma \vdash \Gamma_2$.
Γ should only have symbols common to Γ_1 and Γ_2.

In an infinitary extension of first order logic:

- Γ_1 - First order sentences
- Γ_2 - Universally quantified infinitary disjunctions of atoms

  Note: Can express accessibility properties, e.g.,

$$(\forall x)x = 0 \vee x = s(0) \vee \cdots \vee x = s^n(0) \vee \cdots$$

- Γ - Universally quantified finitary disjunctions of atoms

The interpolant inherits the finitary character of Γ_1 and the structure of Γ_2.

New interpolation results for FOL (I)

Γ_1	Γ_2	Γ	Types of sentences
$\mathcal{FO}$	$\mathcal{P}os$	$\mathcal{P}os$	$\mathcal{FO}$- first order sentences
$\mathcal{P}os^*$	$\mathcal{FO}$	$\mathcal{P}os$	$\mathcal{P}os$- positive sentences (i.e.,built with $\wedge, \vee, \forall, \exists$)
$\mathcal{FO}$	$\forall$	$\forall$	$\forall$- universal sentences (also known as Π^0_0)
$\forall^*$	$\mathcal{FO}$	$\forall$	
$\forall^*$	$\mathcal{P}os$	$\forall\vee$	$\forall\vee$- universally quantified disjunctions of atoms
$\mathcal{FO}$	$\exists^{**}$	$\exists$	$\exists$- existential sentences (also known as Σ^0_0)
$\exists$	$\mathcal{FO}$	$\exists$	
$\mathcal{UH}^*$	$\mathcal{FO}$	$\mathcal{UH}$	$\mathcal{UH}$- universal Horn sentences
$\mathcal{FO}$	$\mathcal{UH}$	$\mathcal{UH}$	

* All function symbols of Γ_1 also occur in Γ_2

** The symmetrical of *

New interpolation results for FOL (II)

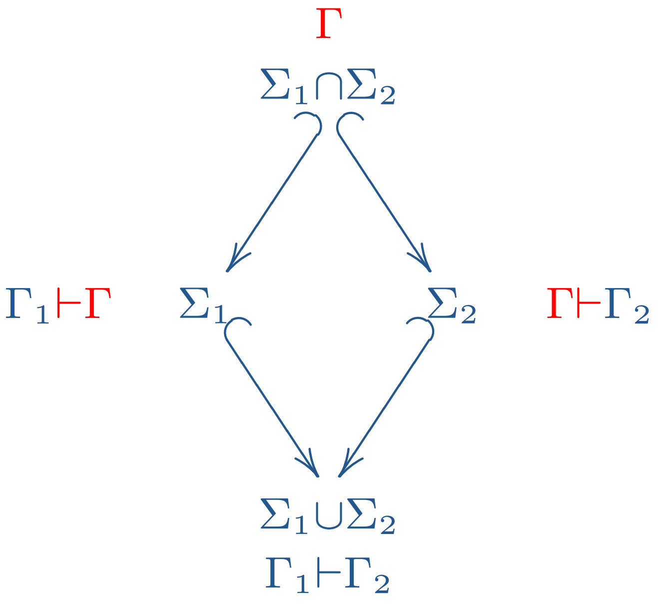
Γ_1	Γ_2	Γ	Types of sentences
$\mathcal{UH}$	$\mathcal{UA}$	$\mathcal{UA}$	$\mathcal{UH}$- universal Horn sentences
$\mathcal{UA}^*$	$\mathcal{FO}$	$\mathcal{UA}$	$\mathcal{UA}$- universally quantified atoms
$\mathcal{UH}^*$	$\mathcal{P}os$	$\mathcal{UA}$	$\mathcal{P}os$- positive (negation-free) sentences
$\mathcal{FO}$	$\forall\vee$	$\forall\vee$	$\forall\vee$- universally quantified disjunctions of atoms
$\forall\vee^*$	$\mathcal{FO}$	$\forall\vee$	
$\mathcal{UH}_\infty$	$\mathcal{UA}$	$\mathcal{UA}$	
$\mathcal{UH}_\infty{}^*$	$\mathcal{FO}_\infty$	$\mathcal{UH}_\infty$	
$\mathcal{FO}_\infty$	$\forall\vee_\infty$	$\forall\vee_\infty$	
$\forall\vee_\infty{}^*$	$\mathcal{FO}_\infty$	$\forall\vee_\infty$	
$\mathcal{FO}$	$\forall\vee_\infty$	$\forall\vee$	

* All function symbols of Γ_1 also occur in Γ_2

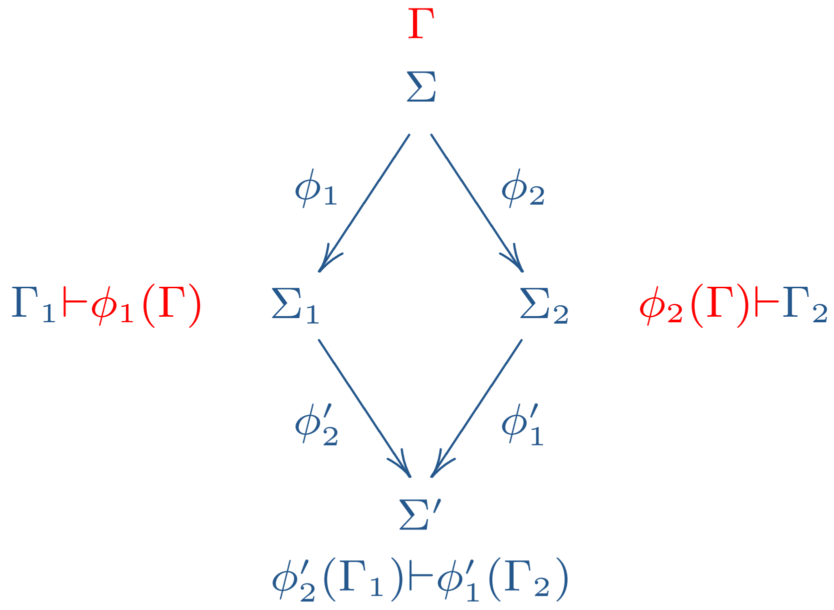
Generalized Craig Interpolation Property

Framework: algebraic specification, with language translations (signature morphisms) that might identify items

Union/intersection square



Arbitrary pushout [Tarlecki, 1986]



Generalized Craig Interpolation Property

- Our results are formulated in this more general framework
- Relax requirements on types of signature morphisms
- Example (many-sorted FOL):
  - [Borzyszkowski, 2000] FOL Interpolation holds if both signature morphisms are injective on sorts
  - Instance of our result: it is enough that (either) one of them be injective on sorts

Conclusions

- Shifted the focus from the syntactic, to a semantic view of interpolation
- Axiomatizability results yield interpolation results
- New interpolation results for various sub-FOL logics
- Combined-logic interpolation: the interpolant takes advantage of the particular form of each side
- Further insight into the issue of signature squares having the interpolation property
- Future work:
  - Extend our technique to other, less conventional logics
  - Try to combine our semantic approach, which provides plenty of information of the interpolant's type, with more constructive approaches