CS522 - Programming Language Semantics

Some Category Theory

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Category theory appeared for at least two important reasons:

1. to capture general concepts and principles from various particular instances of mathematics; and
2. to eliminate some of the foundational concerns, especially those related to set theory.

While the usefulness of category theory is still debatable in mathematics, there is no doubt that category theory is a very powerful language to express and handle complex computer science concepts. In what follows, we shall use category theory as a means to define some very intuitive models of simply-typed λ-calculus. But first, let us introduce some basic notions of category theory.

Category

A *category* $\mathcal{C}$ consists of:

- A *class of objects* written $|\mathcal{C}|$, or $Ob(\mathcal{C})$. It is called a "class" to reflect the fact that it does not need to obey the constraints of set theory; one can think of a class as something "potentially larger than a set";

- A *set of morphisms*, or *arrows*, for any two objects $A, B \in |\mathcal{C}|$, written $\mathcal{C}(A, B)$. The fact that $f \in \mathcal{C}(A, B)$ is often expressed using the more familiar notation $f : A \to B$. The object A is called the *domain* of f, or its *source*, and B is called the *codomain*, or the *target* of f;

- A special *identity morphism* $1_A : A \to A$ for any object $A \in |\mathcal{C}|$;

- A *composition operator* $\_\,;\_ : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C)$ for any (not necessarily distinct) objects $A, B, C \in |\mathcal{C}|$, with the following properties:

- *(identity)* $1_A; f = f; 1_B = f$ for any $A, B \in |\mathcal{C}|$ and any $f : A \to B$, and
- *(associativity)* $(f; g); h = (f; g); h$ for any $f : A \to B$, $g : B \to C$, and $h : C \to D$.

Categories are everywhere. For example:

- $\boldsymbol{Set}$ is the category whose objects are sets and whose morphisms are the usual functions;
- $\boldsymbol{Set}^{inj}$ is the category whose objects are sets and whose morphisms are the injective functions;
- $\boldsymbol{Set}^{surj}$ is the category whose objects are sets and whose morphisms are the surjective functions.

Exercise 1 *Prove that the above are indeed categories.*

Note that it may be the case that there are no morphisms between some given objects. For example, there is no morphism in $\boldsymbol{Set}^{inj}(\{1,2\},\{1\})$. Let us discuss several other categories:

- $\boldsymbol{Mon}$ is the category of monoids, i.e., structures $(M, \_\cdot\_ : M \times M \to M, e \in M)$ with $\_\cdot\_$ associative and identity e, with structure preserving functions, i.e., functions $f : M \to N$ such that $f(a \cdot_M b) = f(a) \cdot_N f(b)$ and $f(e_M) = e_N$, as morphisms;

- $\boldsymbol{Grp}$ is the category of groups and morphisms of groups;

- $\boldsymbol{Poset}$ is the category of partially ordered sets and monotone functions between them;

- $\boldsymbol{Real}^{\leq}$ is the category whose objects are the real numbers and whose morphisms are given by the "$\leq$" relation: $p \to q$ iff $p \leq q$.

Exercise 2 *Show that the above are categories.*

Let us continue the discussion on categorical concepts. Given morphisms $f : A \to B$ and $g : B \to A$, one can obtain the morphisms $f; g : A \to A$ and $g; f : B \to A$. If these morphisms are the identities on A and B, respectively, the morphisms f and g are called *isomorphisms* and the objects A and B are said to be *isomorphic*. The notation $A \simeq B$ is often used to denote that A and B are isomorphic objects.

Diagrams

A *diagram* in a category $\mathcal{C}$ is a directed graph whose nodes are objects and whose arrows are morphisms in $\mathcal{C}$. Formally, a diagram consists of a pair of mappings $d : Nodes \to |\mathcal{C}|$ and $d : Arrows \to \mathcal{C}$, written compactly $d : (Nodes, Arrows) \to \mathcal{C}$, where $(Nodes, Arrows)$ is some (not necessarily finite) labeled digraph (i.e., directed, or oriented, graph), such that for any $\alpha : i \to j$ in $Arrows$, $d(\alpha)$ is a morphism $d(i) \to d(j)$ in $\mathcal{C}$.

To simplify writing, we draw diagrams directly as digraphs and do not specify the mapping explicitly. For example, the following nine figures represent are diagrams:

$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "h"'] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd}$	$A \xrightarrow{f} B$	$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "g"] \\ C \arrow[r, "v"'] & D \end{tikzcd}$
$\begin{tikzcd} & B \\ A \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & C \end{tikzcd}$	$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \end{tikzcd}$	$\begin{tikzcd} A \\ B \end{tikzcd}$
$\begin{tikzcd} & & A \arrow[dll] \arrow[dl] \arrow[r, phantom, "\dots"] \arrow[dr] & & \\ A_1 & A_2 & \dots & A_n & \dots \end{tikzcd}$	$\dots \rightarrow A_n \rightarrow \dots \rightarrow A_1$	$\begin{tikzcd} & A_1 \arrow[ddr, "g_1"] & \\ & A_2 \arrow[dr, "g_2"] & \\ A \arrow[uur, "f_1"] \arrow[ur, "f_2"] \arrow[dr, "f_n"'] & \vdots & A' \\ & A_n \arrow[ur, "g_n"'] & \end{tikzcd}$

A diagram is said to *commute*, or is a *commutative diagram*, iff any two paths between any two nodes correspond to equal morphisms, where path concatenation is interpreted as morphism composition. For example, the top-left diagram commutes iff $f;g = h$; the top-right diagram commutes iff $f;g = u;v$; the middle diagram commutes iff $f = g$; the bottom-right diagram commutes iff $f_i;g_i = f_j;g_j$ for all numbers $1 \le i,j \le n$.

An interesting diagram in the category $\boldsymbol{Real}^{\le}$ is that of positive numbers, with an arrow $p \to q$ iff $p \le q$.

Unless explicitly stated differently, from now on we assume all the diagrams that we draw to be commutative.

Cones and Limits

Given a diagram $d : (Nodes, Arrows) \to \mathcal{C}$, a *cone* of d is a pair $(C, \{\gamma_i\}_{i \in Nodes}$, where C is an object in $|\mathcal{C}|$ and $\gamma_i : C \to d(i)$ are morphisms in $\mathcal{C}$ with the property that $\gamma_i; d(\alpha) = \gamma_j$ for any $\alpha : i \to j$ in $Arrows$:

$$\begin{tikzcd}
 & d(i) \arrow[dd, "d(\alpha)"] \\
C \arrow[ur, "\gamma_i"] \arrow[dr, "\gamma_i"'] & \\
 & d(j)
\end{tikzcd}$$

In other words, all the diagrams formed by the cone with any edge in the diagram commute. The terminology of “cone” probably comes from graphical resemblance with the 3D figure obtained when one regards the diagram as a disc and C as a point above the plane of the disc, which is connected to every point in the diagram.

But, of course, there can be all types of diagrams of all kinds of different "shapes".

Let us next discuss some examples of cones:

- a cone of a diagram containing just one object A and no morphism is any object C together with some morphism $C \to A$ (hence there is a one-to-one correspondence between morphisms of target A and cones of A);
- a cone of a diagram containing two disconnected objects A, B is any object C together with morphisms $f : C \to A$, $g : C \to B$;
- a cone of an empty diagram is any object C (the existence of a morphism from C to any object in the diagram is vacuously fulfilled);
- a cone of a diagram consisting of just a morphism $f : A \to B$ is an object C together with a morphism $g : C \to A$ (the other

morphism of the cone, say $h : C \to B$, is uniquely determined as $g; f$);

- a cone of the diagram of positive real numbers in $\boldsymbol{Real}^{\leq}$ is uniquely determined by any negative number or zero (because these numbers are smaller than or equal to any positive real number); moreover, diagrams in $\boldsymbol{Real}^{\leq}$ admitting cones are precisely those subsets of real numbers which have lower bounds.

A *limit* of a diagram $d : (Nodes, Arrows) \to \mathcal{C}$ is a "maximal" cone d. Formally, a limit of d is a cone $(L, \{\delta_i\}_{i \in Nodes})$ such that for any other cone $(C, \{\gamma_i\}_{i \in Nodes})$ of d, there is a *unique* morphism from

C to L, say $h : C \to L$, such that $h;\delta_i = \gamma_i$ for all $i \in Nodes$:

$$\begin{tikzcd}
 & & d(i) \arrow[dd, "d(\alpha)"] \\
C \arrow[urr, "\gamma_i"] \arrow[drr, "\gamma_j"'] \arrow[r, "\exists! h"] & L \arrow[ur, "\delta_i"] \arrow[dr, "\delta_j"] & \\
 & & d(j)
\end{tikzcd}$$

Exercise 3 *Any two limits of a diagram are isomorphic.*

Because of this, we say that limits are taken "up-to-isomorphism".

Let us next discuss some examples of limits:

- a limit of a diagram containing just one object A and no morphism is any object L that is isomorphic to A (the isomorphism is part of the limit);

- a limit of a diagram containing two disconnected objects A, B is called a *product* of A and B, and is usually written $(A \times B, \pi_A, \pi_B)$, or even more simply just $A \times B$ and the two projections π_A and π_B are understood - the product $A \times B$ has therefore the property that for any object C and morphisms $f : C \to A$ and $g : C \to B$, there is a *unique* morphism, usually written $\langle f, g\rangle : C \to A \times B$, such that $\langle f, g\rangle ; \pi_A = f$ and $\langle f, g\rangle ; \pi_B = g$:

$$
\begin{tikzcd}
 & & A \\
C \arrow[urr, "f"] \arrow[r, "\langle f,g\rangle"] \arrow[drr, "g"'] & A \times B \arrow[ur, "\pi_A"'] \arrow[dr, "\pi_B"] & \\
 & & B
\end{tikzcd}
$$

- A limit of an empty diagram is called a *final* object of the category $\mathcal{C}$, usually denoted $\star$. Recall that a cone of an empty diagram was any object in $C \in |\mathcal{C}|$. Therefore, final objects $\star$ have the property that for any object $C \in \mathcal{C}$ there is a unique morphism from C to $\star$, usually denoted by $!_C : C \to \star$;

- A limit of a diagram consisting of just a morphism $f : A \to B$ is an object isomorphic to A;

- The limit of the diagram of positive real numbers in $\boldsymbol{Real}^{\leq}$ is the number 0, together with the corresponding "less than" morphisms to any positive number. Moreover, any diagram in $\boldsymbol{Real}^{\leq}$ consisting of a bounded set of numbers admits a limit, which is the infimum of the family; if the diagram is a (countable) decreasing sequence, then this limit is precisely the limit from mathematical analysis (this is perhaps where the name "limit" comes from).

Products

Products will play an important role in our subsequent developments. Therefore, we investigate them in slightly more depth here.

Note first that in particular instances of $\mathcal{C}$, for example sets and functions, products are nothing but the usual cartesian products, consisting of pairs of elements, one in the first component and one in the second. Also, the final objects are typically one-element structures.

Exercise 4 *Explain why in* ***Set****, the product of an empty set of sets is a one-element set.*

Given two morphisms $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$, note that there is a *unique* morphism, written $f_1 \times f_2 : A_1 \times A_2 \to B_1 \times B_2$, such that the following diagram commutes:

$$
\begin{tikzcd}
 & A_1 \arrow[rr, "f_1"] & & B_1 \\
A_1 \times A_2 \arrow[ur, "\pi_{A_1}"] \arrow[dr, "\pi_{A_2}"'] \arrow[rr, "f_1 \times f_2"] & & B_1 \times B_2 \arrow[ur, "\pi_{B_1}"] \arrow[dr, "\pi_{B_2}"'] & \\
 & A_2 \arrow[rr, "f_2"'] & & B_2
\end{tikzcd}
$$

Exercise 5 *Show that* $A \times B \simeq B \times A$ *for any* $A, B \in |\mathcal{C}|$.

Exercise 6 *Why the morphism* $f_1 \times f_2$ *exists and is unique?*

Exercise 7 *Show that* $A \simeq \star \times A$ *for any* $A \in |\mathcal{C}|$.

Exponentials

From now on we assume that our categories admit finite products, i.e., limits of finite diagrams of disconnected objects. In particular, the categories are assumed to have final objects.

Given two objects $B, C \in |\mathcal{C}|$, an *exponential of B and C* is an object denoted C^B together with a morphism $app^{B,C} : C^B \times B \to C$ such that for any $f : A \times B \to C$, there is a *unique* $g : A \to C^B$ such that $(g \times 1_B); app^{B,C} = f$:

$$
\begin{tikzcd}
A \times B \arrow[rr, "f"] \arrow[dr, "g\times 1_B"'] & & C \\
& C^B \times B \arrow[ur, "app^{B,C}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[rr, "g"] & & C^B \\
A \times B \arrow[ur] \arrow[dr] \arrow[rr, "g\times 1_B"] & & C^B \times B \arrow[ur] \arrow[dr] & \\
& B \arrow[rr, "1_B"'] & & B
\end{tikzcd}
$$

Proposition 1 *If an exponential* C^B *of* B *and* C *exists in* $\mathcal{C}$*, then there is a one-to-one correspondence between the sets of morphisms* $\mathcal{C}(A \times B, C)$ *and* $\mathcal{C}(A, C^B)$*. The two components of this bijection, inverse to each other, are written:*

$$\begin{tikzcd} \mathcal{C}(A \times B, C) \arrow[r, bend left, "curry"] & \mathcal{C}(A, C^B) \arrow[l, bend left, "uncurry"] \end{tikzcd}$$

Proof. Let us first define the functions *curry* and *uncurry*. For any $f : A \times B \to C$, let $curry(f)$ be the unique morphism $g : A \to C^B$ given by the definition of the exponential, with the property that $(g \times 1_B); app^{B,C} = f$. Conversely, for any $g : A \to C^B$, let $uncurry(g)$ be the morphism $(g \times 1_B); app^{B,C} : A \times B \to C$. All we need to prove is that for any $f : A \times B \to C$ and $g : A \to C^B$, it is the case that $uncurry(curry(f)) = f$ and $curry(uncurry(g)) = g$. The first is equivalent to $(curry(f) \times 1_B); app^{B,C} = f$, which is immediate by the definition of *curry*, while the second follows by the unicity of g with the property that $(g \times 1_B); app^{B,C} = f$, where f is *uncurry*(g). $\square$

Exercise 8 *Prove that* $\mathcal{C}(A, B) \simeq \mathcal{C}(\star, B^A)$ *whenever the exponential of* A *and* B *exists in* $\mathcal{C}$*.*

A category $\mathcal{C}$ which admits finite products and exponentials for any two objects is called *cartesian closed*. For notational simplicity, a cartesian closed category is called a *CCC*.